# Lecture 3: Fluid Models For Tokamak Plasmas

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*Questions To Be Addressed:*

**1) What extended MHD model results from moment equations?**

**2) Macrostability constraints, flux surfaces, currents in tokamaks?**

**3) What are the complete tokamak plasma transport equations?**

*Outline:*

- **Extended MHD model obtained from fluid moment equations**
- **Ideal MHD model, macroinstabilities and consequences**
- **Axisymmetric tokamak magnetic field geometry**
- **Neoclassical stress closures for tokamak plasmas**
- **Plasma currents and flows in tokamaks**
- **Comprehensive tokamak plasma transport equations**

# Extended MHD Model Uses Fluid Moment Equations

- Recall the species $s$ *fundamental fluid moment equations*:

*density* $$(\partial/\partial t + \vec{V}_s\cdot\vec{\nabla})\, n_s = -n_s\vec{\nabla}\cdot\vec{V}_s + S_{ns},$$

*mom.* $$m_s n_s(\partial/\partial t + \vec{V}_s\cdot\vec{\nabla})\,\vec{V}_s = n_s q_s(\vec{E} + \vec{V}_s\times\vec{B}) - \vec{\nabla}p_s - \vec{\nabla}\cdot\overleftrightarrow{\pi}_s + \vec{R}_s + \vec{S}_{\rm ps},$$

*entropy* $$(\partial/\partial t + \vec{V}_s\cdot\vec{\nabla})\,{\rm s}_{{\rm M}s} = \dot{\rm s}_{{\rm M}s} \equiv (-\vec{\nabla}\cdot\vec{q}_s - \overleftrightarrow{\pi}_s : \vec{\nabla}\vec{V}_s + Q_s + S_{\mathcal{E}s})/p_s.$$

- Extended MHD equations are obtained by summing fluid equations over $e, i$ species using the definitions (assuming $|\vec{V}_i| \ll v_{Ti}$)

| | |
|---|---|
| mass density (kg/m$^3$) | $\rho_m \equiv \sum_s m_s n_s = m_e n_e + m_i n_i \simeq m_i n_i$ |
| mass flow velocity (m/s) | $\vec{V} \equiv \sum_s m_s n_s \vec{V}_s/\rho_m \simeq \vec{V}_i$ |
| current density (A/m$^2$) | $\vec{J} \equiv \sum_s n_s q_s \vec{V}_s = -n_e e\,(\vec{V}_e - \vec{V}_i)$ |
| plasma pressure (N/m$^2$) | $P \equiv \sum_s p_s = p_e + p_i$ |
| stress tensor (N/m$^2$) | $\overleftrightarrow{\Pi} \equiv \sum_s \overleftrightarrow{\pi}_s \simeq \overleftrightarrow{\pi}_i.$ |

# Extended MHD Model Includes Ideal MHD And The Dissipative Effects Of Collisional And Closure Moments

- The extended MHD equations for a magnetized plasma and the associated electric and magnetic fields are thus (neglecting sources)

***Extended MHD plasma description*** (for ideal MHD $\vec{R}_e$, $\overleftrightarrow{\Pi}$, $\overleftrightarrow{\pi}_e$, $\sum_s \dot{s}_{Ms} \to 0$):

$$
\begin{aligned}
\text{mass density} \quad & (\partial/\partial t + \vec{V}\cdot\vec{\nabla})\,\rho_m = -\rho_m \vec{\nabla}\cdot\vec{V}, \\
\text{charge continuity} \quad & \vec{\nabla}\cdot\vec{J} = 0, \\
\text{momentum} \quad & \rho_m\,(\partial/\partial t + \vec{V}\cdot\vec{\nabla})\,\vec{V} = \vec{J}\times\vec{B} - \vec{\nabla}P - \vec{\nabla}\cdot\overleftrightarrow{\Pi}, \\
\text{Ohm's law} \quad & \vec{E} = -\vec{V}\times\vec{B} + \vec{R}_e/n_e e + (\vec{J}\times\vec{B} - \vec{\nabla}p_e - \vec{\nabla}\cdot\overleftrightarrow{\pi}_e)/n_e e, \\
\text{equation of state} \quad & (\partial/\partial t + \vec{V}\cdot\vec{\nabla})\ln(P/\rho_m^{5/3}) = \textstyle\sum_s \dot{s}_{Ms}.
\end{aligned}
$$

***Maxwell Equations for extended MHD*** (no Gauss' law, $\vec{E}$ from Ohm's law):

$$
\begin{aligned}
\text{Faraday's law} \quad & \partial\vec{B}/\partial t = -\vec{\nabla}\times\vec{E}, \\
\text{no magnetic monopoles} \quad & \vec{\nabla}\cdot\vec{B} = 0, \\
\text{nonrelativistic Ampere's law} \quad & \vec{J} = \vec{\nabla}\times\vec{B}/\mu_0.
\end{aligned}
$$

# Ideal MHD Model Has Some Special Properties

- Collisional effects are negligible unless $\omega \lesssim \nu_{\rm eff}$ at resonances.

- Faraday's law plus $\vec{E} = -\vec{V}\times\vec{B}$ produce the frozen flux theorem:
  $\dfrac{\partial \vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E} = \vec{\nabla}\times(\vec{V}\times\vec{B}) \implies \dfrac{d\Psi}{dt} = \dfrac{d}{dt}\iint_S \vec{B}\cdot d\vec{S} = 0,$ which causes $\vec{B}$ to be advected with $\vec{V}$ perturbations (as in hydrodynamics Kelvin theorem).

- There are various types of ideal MHD Alfvén waves:
  compressional, $\omega = k_\perp\sqrt{c_{\rm A}^2 + c_{\rm S}^2} \simeq k_\perp c_{\rm A}$ — "fast" compressible waves $\perp$ to $\vec{B}$,
  shear/torsional, $\omega = k_\| c_{\rm A}$ — incompressible waves propagating $\|$ to $\vec{B}$,
  parallel sound, $\omega = k_\| c_{\rm S}$ — compressible waves propagating $\|$ to $\vec{B}$,
  in which Alfvén speed is $c_{\rm A} \equiv B/\sqrt{\mu_0\rho_m}$ and sound speed $c_{\rm S} \equiv \sqrt{(5/3)P/\rho_m}$.

- In tokamak plasmas which have $c_{\rm A} \propto c_{\rm S}/\sqrt{\beta} \gg c_{\rm S}$,
  compressional waves enforce equilibrium radial plasma & ion force balances,
  while shear, sound waves can become unstable $\implies$ operational constraints.

# Ideal MHD Provides Tokamak Plasma Constraints

- **Stable compressional Alfvén waves enforce equilibrium radial force balance on very short time scales ($\bar{a}/c_A \sim 10^{-7} - 10^{-6}$ s) and yield**

  **ideal MHD equilibrium equations:** $\vec{J}\times\vec{B} = \vec{\nabla}P, \quad \vec{J} = \vec{\nabla}\times\vec{B}/\mu_0, \quad \vec{\nabla}\cdot\vec{B} = 0.$

- **If the shear Alfvén or sound waves become unstable,**

  **they grow on very fast time scales ($R/c_A \sim 10^{-5} - 10^{-6}$ s), and**

  **usually lead to virulent global instabilities and hence plasma "disruptions."**

- **Stability criteria for avoiding these ideal MHD macroinstabilities provide limits on parameter regimes in which tokamaks operate:**

  **sound wave stability,** $\beta \equiv \dfrac{P}{B^2/2\mu_0} \lesssim \dfrac{a}{Rq} \sim 0.1$ **(analogous to Rayleigh-Taylor),**

  **shear Alfvén stability,** $q \simeq \dfrac{aB_t}{RB_p} \geq 1$ **(Kruskal-Shafranov criterion, kink modes).**

- **Further tokamak analyses assume these ideal MHD stability criteria are satisfied so virulent macroinstabilities are circumvented.**

# Tokamaks Have Axisymmetric MHD Equilibrium Magnetic Flux Surfaces And Coordinate Systems

- **Key coordinates are major radius $R$, poloidal angle $\theta$ and toroidal (axisymmetry) angle $\zeta$ and poloidal magnetic flux $\psi_{\rm p}$.**

- **Poloidal magnetic field $\vec{B}_{\rm p}$ is defined via poloidal magnetic flux:**

$$\psi_{\rm p} = \frac{1}{2\pi}\iint d\vec{S}_\theta\cdot\vec{B}_{\rm p} = \frac{1}{2\pi}\iint d\vec{S}_\theta\cdot\vec{\nabla}\times\vec{A}_{\rm t} = \frac{1}{2\pi}\int d\vec{\ell}\cdot\vec{A}_{\rm t} = -RA_{\rm t}, \quad \text{which yields}$$

$$\vec{B}_{\rm p} \equiv \vec{\nabla}\times\vec{A}_{\rm t} = \vec{\nabla}\times(-\psi_{\rm p}\vec{\nabla}\zeta) = \vec{\nabla}\zeta\times\vec{\nabla}\psi_{\rm p}.$$

- **Toroidal magnetic field $\vec{B}_{\rm t}$ is axisymmetric so since $|\vec{\nabla}\zeta| = 1/R$,**

$$\vec{B}_{\rm t} = RB_{\rm t}\,\vec{\nabla}\zeta = I\,\vec{\nabla}\zeta, \quad \text{in which } I(\psi_{\rm p},\theta) = RB_{\rm t}.$$

- **The axisymmetric ($\partial/\partial\zeta = 0$) helical magnetic field in a tokamak is thus composed of toroidal and poloidal components:**

$$\vec{B} = \vec{B}_{\rm t} + \vec{B}_{\rm p} = I\,\vec{\nabla}\zeta + \vec{\nabla}\zeta\times\vec{\nabla}\psi_{\rm p} = \vec{\nabla}\psi_{\rm p}\times\vec{\nabla}[\,q(\psi_{\rm p})\,\theta - \zeta\,], \quad \text{in which}$$

$$q(\psi_{\rm p}) = \frac{d\zeta}{d\theta} \equiv \frac{\vec{B}_{\rm t}\cdot\vec{\nabla}\zeta}{\vec{B}_{\rm p}\cdot\vec{\nabla}\theta} = \frac{I}{R^2(\vec{B}_0\cdot\vec{\nabla}\theta)} \text{ for a “straight field line” coordinate } \theta.$$

# Poloidal Flux Surfaces Obey Grad-Shafranov Equation

- **The tokamak magnetic field and current density are**

  $$\vec{B} \equiv I\,\vec{\nabla}\zeta + \vec{\nabla}\zeta\times\vec{\nabla}\psi_{\rm p}, \quad \mu_0\vec{J} = \vec{\nabla}\times\vec{B} = \vec{\nabla}I\times\vec{\nabla}\zeta + \vec{\nabla}\zeta\;\Delta^*\psi_{\rm p},$$

  **in which the magnetic differential operator $\Delta^*$ is defined by**

  $$\boxed{\Delta^*\psi_{\rm p} \equiv \frac{1}{|\vec{\nabla}\zeta|^2}\vec{\nabla}\cdot(|\vec{\nabla}\zeta|^2\vec{\nabla}\psi_{\rm p}) = R^2\,\vec{\nabla}\cdot\frac{\vec{\nabla}\psi_{\rm p}}{R^2} = \frac{\partial^2\psi_{\rm p}}{\partial R^2} - \frac{1}{R}\frac{\partial\psi_{\rm p}}{\partial R} + \frac{\partial^2\psi_{\rm p}}{\partial Z^2}.}$$

- **Ideal MHD equilibrium, radial force balance $\vec{J}\times\vec{B} = \vec{\nabla}P$ yields**

  $$\vec{B}\cdot\vec{\nabla}P = 0 \implies P = P(\psi_{\rm p}),$$

  $$\vec{J}\cdot\vec{\nabla}P = 0 \implies (dP/d\psi_{\rm p})\,(\vec{J}\cdot\vec{\nabla}\psi_{\rm p}) = 0 \implies \partial I/\partial\theta = 0 \implies I = I(\psi_{\rm p}),$$

  $$\vec{\nabla}\psi_{\rm p}\cdot(\vec{J}\times\vec{B} - \vec{\nabla}P) = 0 \implies \boxed{\Delta^*\psi_{\rm p} = -\,\mu_0 R^2\frac{dP(\psi_{\rm p})}{d\psi_{\rm p}} - I(\psi_{\rm p})\,\frac{dI(\psi_{\rm p})}{d\psi_{\rm p}},}$$

  **which is called the Grad-Shafranov equation.**

- **For specified functions $P(\psi_{\rm p})$ and $I(\psi_{\rm p})$, this is a nonlinear elliptic equation for $\psi_{\rm p}(R, Z)$ that is usually solved numerically.**

# Flux Surfaces Shift Outward As Plasma $\beta_{\rm p}$ Increases

- **Poloidal flux surfaces in a circular fixed boundary tokamak**[1]

$$\text{have key variables } A \equiv \frac{\text{major radius}}{\text{minor radius}} = \frac{R}{a} \text{ (aspect ratio)}, \quad \beta_{\rm p} \equiv \frac{P}{B_{\rm p}^2/2\mu_0}.$$

[1] J.D. Callen and R.A. Dory, "Magnetohydrodynamic Equilibria in Sharply Curved Axisymmetric Devices," Phys. Fluids **15**, 1523 (1972).

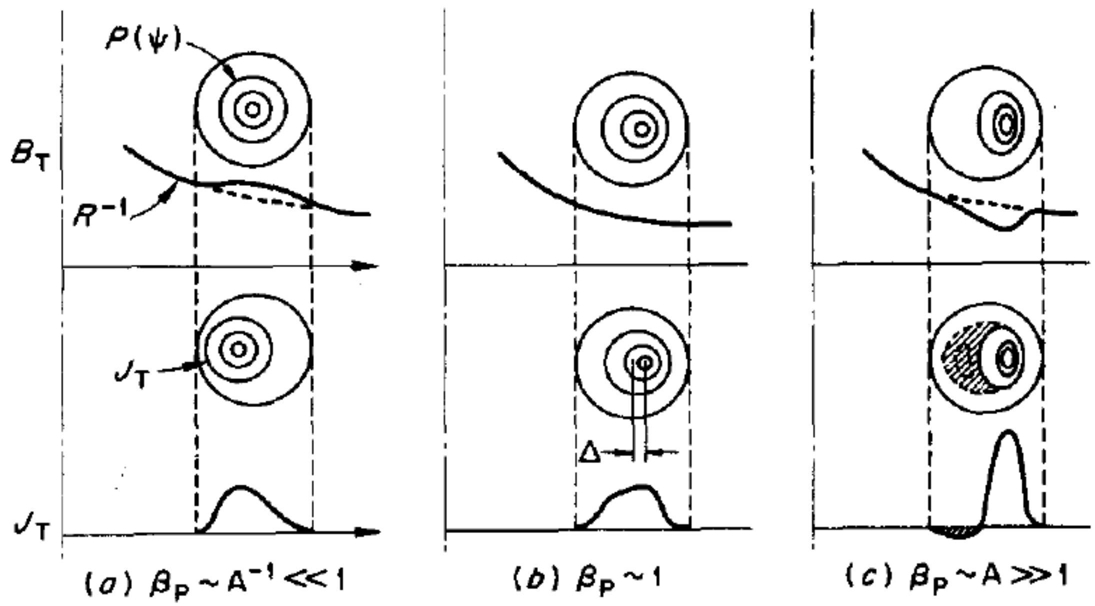


Figure 1: **Schematic illustration of $\psi_{\rm p}$ flux surfaces showing variation of $B_{\rm t}$ with $R$, surfaces of constant toroidal current $J_{\rm t}$, and variation of $J_{\rm t}$ with $R$ on the mid-plane. Shading in (c) indicates a region with reversed current. The parameter $\Delta$ measures outward nesting of flux surfaces and is called the Shafranov shift.**

# DIII-D Tokamak Experiment Has Divertor Separatrix

- **DIII-D Parameters:**

  major radius 1.7 m,

  minor radius 0.6 m,

  aspect ratio 2.8.

- **Flux surface geometry is**

  a two-dimensional (2-D) magnetic geometry,

  has a free boundary,

  and imbedded divertor magnetic separatrix,

  for which the Grad-Shafranov equation is solved numerically.

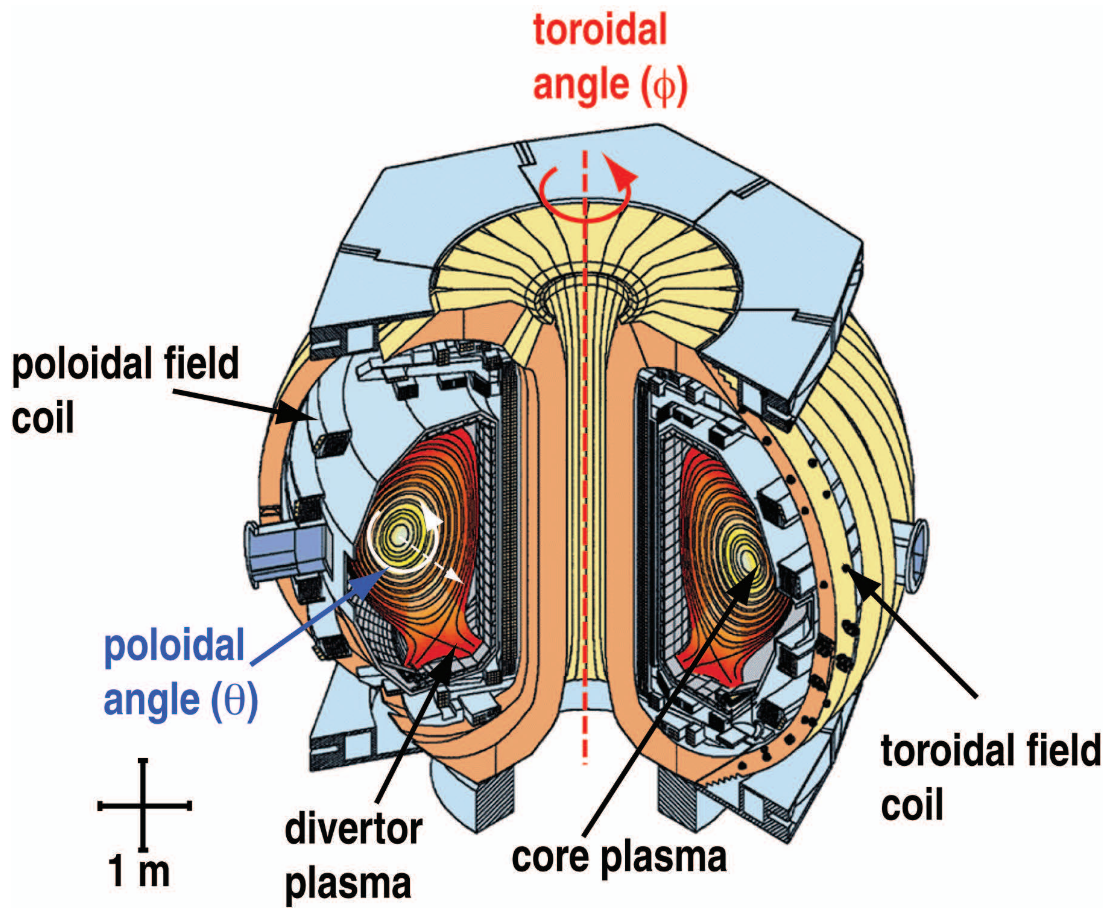


Figure 2: **DIII-D experiment at GA in San Diego, CA (`http://fusion.gat.com/global/DIII-D`) is a U.S. national facility that has been operating since late 1980s, continually adding diagnostics & hardware.**

# Collisional Dissipative Effects Are Important For $t > 1/\nu$

- **Resistivity effects reconnect (or tear) magnetic field lines in thin singular layers around low order rational surfaces where $q(\psi_{\rm p}) = m/n$ on which the helical magnetic field lines close on themselves.**

- **This reconnection violates the frozen flux theorem of ideal MHD and can allow slowly growing, radially isolated tearing-type (classical $\vec{\nabla} J_{\rm t}$-driven and neoclassical $\vec{\nabla} P$-driven) macroinstabilities.**

- **These modes can cause magnetic island topologies to develop in the plasma which sometimes continue to grow and violently disrupt plasma confinement, i.e., lead to a plasma "disruption."**

- **When such deleterious macroinstabilities are controlled, the equilibrium extended MHD equations yield prescriptions for the first order (in small gyroradius expansion) equilibrium and perturbed flows & currents (and hence Ohm's law) on magnetic flux surfaces.**

- **Key closure for low collisionality tokamaks is viscous stress $\overleftrightarrow{\pi}_s$.**

# Low Collisionality Viscous Stresses Are Different

- **For a physical analogy, consider flow of a neutral ($_n$) fluid down a "bumpy cylinder pipe" of radius $a$ and axial periodicity length $L_\parallel$:**

  **for short collision lengths (i.e., $\lambda_n \ll a, L_\parallel$) the viscous diffusivity is isotropic and $\mu \sim \nu_n \lambda_n^2 = v_{Tn}^2/\nu_n$ — momentum diffuses to walls at rate $\nu_n \lambda_n^2/a^2 \ll \nu_n$;**

  **however, for long collision lengths (i.e., $\lambda_n \gg L_\parallel$) if the neutrals were held at the same radius (as charged particles are by $\vec{B}$), the parallel (axial direction) viscous diffusivity would be $\mu_z \sim \nu_n L_\parallel^2$ — axial momentum would be relaxed by collisions with the $L_\parallel$ bumps instead of over the collision length $\lambda_n$. Then the momentum relation rate would be $\mu_z |(\partial^2 V_z/\partial z^2)/V_z| \sim \nu_n L_\parallel^2/L_z^2 \sim \nu_n$.**

- **Low collisionality tokamak plasmas have long collision lengths $\lambda_s \equiv v_{Ts}/\nu_s$ compared to the $L_\parallel \simeq Rq$ for variations of $B = |\vec{B}|$ along the $\vec{B} = \vec{B}_\text{t} + \vec{B}_\text{p}$ helical magnetic field lines.**

- **Key issue for determining parallel flows and electrical resistivity in tokamak fusion plasmas is determination of the parallel viscous stresses $\overleftrightarrow{\pi}_{s\parallel}$ and resultant forces in the low collisionality regime.**

# Consider Collisional Stresses In A Magnetized Plasma

- Collisional Braginskii viscous stresses are defined relative to $\vec{B}$ direction:

$$\overleftrightarrow{\pi} = \overleftrightarrow{\pi}_{\parallel} + \overleftrightarrow{\pi}_{\wedge} + \overleftrightarrow{\pi}_{\perp}, \quad \text{parallel, cross (gyroviscous) and perpendicular stresses.}$$

- For strongly magnetized ($\omega_c \gg 1/\nu$) toroidal plasmas of fusion interest a small gyroradius expansion is usually appropriate: $\varrho_* \equiv \varrho/a \ll 1$.

- For arbitrary $\vec{V}$, the characteristic scalings of the parallel, cross and perpendicular stresses can be written schematically for $Rq \gtrsim \lambda \gtrsim a$ as

$$\overleftrightarrow{\pi}_{\parallel} \sim \nu\lambda^2\, \vec{\nabla}_{\parallel}\vec{V}, \quad \overleftrightarrow{\pi}_{\wedge} \sim \nu\varrho\,\lambda\, \vec{B}\times\vec{\nabla}\vec{V}/B \sim \varrho_*\, \overleftrightarrow{\pi}_{\parallel}, \quad \overleftrightarrow{\pi}_{\perp} \sim \nu\varrho^2\, \vec{\nabla}_{\perp}\vec{V} \sim \varrho_*^2\, \overleftrightarrow{\pi}_{\parallel}.$$

- Thus, the parallel viscous stress $\overleftrightarrow{\pi}_{\parallel}$ is dominant in small gyroradius, magnetized toroidal plasmas. We concentrate on it. The $\overleftrightarrow{\pi}_{\wedge}$ and $\overleftrightarrow{\pi}_{\perp}$ are changed less.

- The parallel viscous stresses for electrons and ions were originally written by Braginskii for each species in the form ($z$ here is coordinate along $\vec{B}$, $Z_i = 1$)

$$\overleftrightarrow{\pi}_{\parallel} = -\,\eta_0\, W_{zz}\, \hat{\vec{e}}_z\hat{\vec{e}}_z, \quad W_{zz} \equiv 2\,\frac{\partial V_z}{\partial z} - \frac{2}{3}\,(\vec{\nabla}\cdot\vec{V}), \quad \eta_0^i = 0.48\, n_i m_i \frac{v_{Ti}^2}{\nu_i}, \quad \eta_0^e = 0.37\, n_e m_e \frac{v_{Te}^2}{\nu_e}.$$

- But this is not valid for low collisionality tokamak plasmas where $\lambda_e \equiv v_{Te}/\nu_e \gg L_{\parallel}$.

# Stress Tensor Has A General Magnetic Field Geometry Form

- Braginskii viscous force due to CGL form for parallel stresses is

$$\overleftrightarrow{\pi}_\parallel \equiv \pi_\parallel\left(\frac{\vec{B}\vec{B}}{B^2} - \frac{\overleftrightarrow{\mathbf{I}}}{3}\right), \qquad \pi_\parallel \equiv -\frac{3}{2}\eta_0\,\frac{\vec{B}\cdot\mathsf{W}\cdot\vec{B}}{B^2}, \qquad \mathsf{W} \equiv \vec{\nabla}\vec{V} + (\vec{\nabla}\vec{V})^{\mathrm{T}} - \frac{2}{3}\overleftrightarrow{\mathbf{I}}(\vec{\nabla}\cdot\vec{V}).$$

- Parallel component of parallel rate of strain tensor has a couple of forms:

$$\boxed{\vec{B}\cdot\mathsf{W}\cdot\vec{B}/2} = B(\vec{B}\cdot\vec{\nabla})(\vec{V}\cdot\vec{B}/B) + [\vec{B}\times(\vec{B}\times\vec{V})]\cdot\vec{\kappa} - (B^2/3)\vec{\nabla}\cdot\vec{V}$$

$$\boxed{= B^2\vec{V}\cdot\vec{\nabla}\ln B + \vec{B}\cdot\vec{\nabla}\times(\vec{V}\times\vec{B}) + (2B^2/3)\vec{\nabla}\cdot\vec{V} - (\vec{B}\cdot\vec{V})(\vec{\nabla}\cdot\vec{B}).}$$

- After fast time scales of compressional Alfvén and sound wave relaxations, $\vec{B}\cdot\vec{\nabla}\times(\vec{V}\times\vec{B}) \simeq 0$, $\vec{\nabla}\cdot\vec{V} = 0$ and $\vec{V}_\perp = (1/B^2)\vec{B}\times\vec{\nabla}f$, the last form yields

$$\pi_\parallel = -3\eta_0(\vec{V}\cdot\vec{\nabla}\ln B) + \Delta\pi_\parallel, \quad \Delta\pi_\parallel \equiv -(3\eta_0/B^3)(\vec{B}\cdot\vec{\nabla}f)[\vec{B}\cdot\vec{\nabla}\times(\vec{B}/B)] \sim \beta(k_\parallel a).$$

- Viscous force from Braginskii viscous stress is [$\vec{\kappa} \equiv (\hat{\mathrm{b}}\cdot\vec{\nabla})\hat{\mathrm{b}}$ is curvature vector]

$$\vec{\nabla}\cdot\overleftrightarrow{\pi}_\parallel = \pi_\parallel\,[\vec{\kappa} - \vec{B}(\vec{B}\cdot\vec{\nabla}\ln B)/B^2] + (1/B^2)\vec{B}(\vec{B}\cdot\vec{\nabla})\pi_\parallel - (1/3)\vec{\nabla}\pi_\parallel$$

$$\Longrightarrow \qquad \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_\parallel = -\pi_\parallel\,(\vec{B}\cdot\vec{\nabla}\ln B) + (2/3)(\vec{B}\cdot\vec{\nabla})\pi_\parallel.$$

- Flux-surface-average (FSA), neglect the small $\Delta\pi_\parallel$, and use $\vec{V}\cdot\vec{\nabla}\ln B$ $= (\vec{B}\cdot\vec{\nabla}\ln B)\,U_\theta(\psi_p)$ to obtain "residual" FSA parallel (to $\vec{B}$) viscous force:

$$\boxed{\langle\vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_\parallel\rangle \simeq 3\eta_0\,\langle(\vec{B}\cdot\vec{\nabla}\ln B)^2\rangle\,U_\theta,} \quad \text{in which } U_\theta(\psi_p) \equiv \frac{\vec{V}\cdot\vec{\nabla}\theta}{\vec{B}\cdot\vec{\nabla}\theta} \text{ from } \vec{\nabla}\cdot\vec{V} = 0.$$

## Tokamak Neoclassical Theory Uses 2-D Axisymmetric (A) $\vec{B}$ Field

- **An axisymmetric magnetic field and coordinate system are needed to connect to FSA parallel viscous forces in low $\nu$ "neoclassical" transport theory.**[2,3]

- **The 2-D axisymmetric (A) equilibium magnetic field $\vec{B}_0 \equiv \vec{B}_{\rm t} + \vec{B}_{\rm p}$ has toroidal and poloidal parts. It is written in terms of the poloidal magnetic flux $\psi_{\rm p}$:**

$$\vec{B}_0(\psi_{\rm p}, \theta) = I\,\vec{\nabla}\zeta + \vec{\nabla}\zeta \times \vec{\nabla}\psi_{\rm p} = \vec{\nabla}\psi_{\rm p} \times \vec{\nabla}\,[\,q(\psi_{\rm p})\,\theta - \zeta\,], \quad I(\psi_{\rm p}) \equiv RB_{\rm t}.$$

- **The radial, poloidal straight-field-line, and toroidal axisymmetry coordinates are taken to be $\psi_{\rm p}, \theta, \zeta$ for which the poloidal rotation of a field line per unit toroidal rotation is $d\theta/d\zeta = 1/q(\psi_{\rm p}) \equiv \vec{B}_0 \cdot \vec{\nabla}\theta / \vec{B}_0 \cdot \vec{\nabla}\zeta$.**

- **The Jacobian for transforming from the laboratory ($\vec{x}$) to these (non-orthogonal) curvilinear coordinates is $\sqrt{g} \equiv 1/(\vec{\nabla}\psi_p \cdot \vec{\nabla}\theta \times \vec{\nabla}\zeta) = 1/\vec{B}_0 \cdot \vec{\nabla}\theta = qR^2/I$. The 2-D flux surface average (FSA) of a scalar function $f(\vec{x})$ is defined by**

$$\langle f(\vec{x}) \rangle \equiv \frac{\int_0^{2\pi} d\zeta \int_0^{2\pi} f(\vec{x})\; d\theta / \vec{B}_0 \cdot \vec{\nabla}\theta}{2\pi \int_0^{2\pi} d\theta / \vec{B}_0 \cdot \vec{\nabla}\theta}, \quad \text{flux surface average of } f(\vec{x}).$$

- **The FSA annihilates parallel derivatives of scalar functions: $\langle \vec{B}_0 \cdot \vec{\nabla} f \rangle = 0$.**

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[2] F.L. Hinton and R.D. Hazeltine, "Theory of plsma transport in toroidal confinement systems," Rev. Mod. Phys. **48**, 239 (1976)
[3] S.P. Hirshman and D.J. Sigmar, "Noclassical transport of impurities in tokamak plasmas," Nucl. Fusion **21**, 1079 (1981).

# FSA Neoclassical Parallel Viscous Closures Have Matrix Structure

- **In all collisionality regimes the residual FSA parallel viscous force $\langle \vec{B}_0 \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_\parallel \rangle$ and parallel viscous heat force $\langle \vec{B}_0 \cdot \vec{\nabla} \cdot \overleftrightarrow{\Theta}_\parallel \rangle$ can be written in matrix form:**[3,4]

$$\begin{bmatrix} \langle \vec{B}_0 \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_\parallel \rangle \\ \langle \vec{B}_0 \cdot \vec{\nabla} \cdot \overleftrightarrow{\Theta}_\parallel \rangle \end{bmatrix} = \frac{mn}{\tau} \langle B_0^2 \rangle \, \mathsf{M} \cdot \begin{bmatrix} U_\theta \\ Q_\theta \end{bmatrix}, \quad \frac{1}{\tau_{ss}} \equiv \frac{4}{3\sqrt{\pi}} \frac{4\pi \, n_s q_s^4 \ln \Lambda}{\{4\pi\epsilon_0\}^2 \, m_s^2 \, v_{Ts}^3}, \text{ collision frequency.}$$

- **The matrix of dimensionless viscosity coefficients $\mathsf{M}$ is defined by**

$$\mathsf{M} \equiv \begin{bmatrix} \mu_{00} & \mu_{01} \\ \mu_{01} & \mu_{11} \end{bmatrix} = \nu_{\rm ref} \tau_{ss} \frac{f_t}{f_c} \begin{bmatrix} \hat{K}_{00} & \frac{5}{2}\hat{K}_{00} - \hat{K}_{01} \\ \frac{5}{2}\hat{K}_{00} - \hat{K}_{01} & \hat{K}_{11} - 5\,\hat{K}_{01} + \frac{25}{4}\hat{K}_{00} \end{bmatrix}, \quad \frac{f_t}{f_c} \sim 1.46\sqrt{\epsilon}, \text{ trap frac.}$$

- **The multi-collisionality "total" positive-definite coefficients $\hat{K}_{ij}^{\rm tot}$ are**[4,5]

$$\hat{K}_{ij}^{\rm tot} = \frac{\hat{K}_{ij}^{B}}{\left[1 + \nu_{*s}^{1/2} + 2.92\,\nu_{*s} \hat{K}_{ij}^{B} / \hat{K}_{ij}^{P}\right] \left[1 + 2\hat{K}_{ij}^{P} / (3\,\omega_{ts} \tau_{ss} \hat{K}_{ij}^{PS})\right]}, \quad \nu_{*s} \sim \frac{\nu_s}{\epsilon^{3/2} v_T / R_0 q}.$$

[4]See Ref. [11] supplementary material in J.D. Callen, C.C. Hegna and A.J. Cole, Phys. Plasmas **17**, 056113 (2010), which is available as J.D. Callen, "Viscous Forces Due To Collisional Parallel Stresses For Extended MHD Codes," Report UW-CPTC 09-6R via `http://www.cptc.wisc.edu/Reports.html`.

[5]In retrospect, the $\nu_{*s}^{1/2}$ low collisionality regime boundary layer term added phenomenologically in Ref. 4 and $\hat{K}_{ij}^{\rm tot}$ probably should be omitted.

Table 1: Asymptotic dimensionless viscosity coefficients. In plasmas with impurities the ion charge $Z$ becomes $Z_{\rm eff}$ for electrons and similar modifications occur for ions. In rightmost column $D \equiv (6/5)(2Z^2 + 301/48\sqrt{2} + 89/48)$ is $2{\times}2$ determinant of $\mathsf{G}$.

| regime: | banana ($B$) | plateau ($P$) | Pfirsch-Schlüter ($PS$), Braginskii |
|---|---|---|---|
| $\hat{K}_{00}$ | $[Z + \sqrt{2} - \ln(1+\sqrt{2})]/(\nu_s \tau_{ss})$ | $\sqrt{\pi}$ | $(17Z/4 + 205/48\sqrt{2})/D$ |
| $\hat{K}_{01}$ | $[Z + 1/\sqrt{2}]/(\nu_s \tau_{ss})$ | $3\sqrt{\pi}$ | $(7/2)(23Z/4 + 241/48\sqrt{2})/D$ |
| $\hat{K}_{11}$ | $[2Z + 9/4\sqrt{2}]/(\nu_s \tau_{ss})$ | $12\sqrt{\pi}$ | $(49/4)(33Z/4 + 325/48\sqrt{2})/D$ |

# Poloidal Flow Is Obtained From Plasma $\parallel$ Force Balance

- Summing $\parallel$ force balances (momentum equations) over species yields (neglecting fluctuations and sources here for simplicity)

$$m_i n_{i0} \frac{\partial \langle B_0 V_{i\parallel} \rangle}{\partial t} \simeq -\langle \vec{B}_0 \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_i \rangle.$$

- The poloidal flow is determined mainly by the ion $\parallel$ viscous force:

$$\langle \vec{B} \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_{i\parallel} \rangle \simeq \frac{m_i n_{i0}}{\tau_{ii}} \left[ \mu_{i00} U_{i\theta} + \mu_{i01} \frac{-2}{5 n_i T_i} Q_{i\theta} + \cdots \right] \langle B^2 \rangle, \qquad \mu_{i00}, \mu_{i01} \sim \sqrt{\epsilon}.$$

- For $t > 1/\nu_i \sim 1$ ms, poloidal flow obtained from $\langle \vec{B}_0 \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_{i\parallel} \rangle \simeq 0$ is

$$U^0_{i\theta}(\psi_p) \equiv \frac{\vec{V} \cdot \vec{\nabla} \theta}{\vec{B} \cdot \vec{\nabla} \theta} \simeq -\frac{\mu_{i01}}{\mu_{i00}} \frac{-2}{5 n_i T_i} Q_{i\theta} \simeq k_i \frac{I}{q_i \langle B^2 \rangle} \frac{dT_{i0}}{d\psi_{\rm p}} \quad \Longrightarrow \quad \boxed{V_{\rm p} \simeq \frac{1.17}{q_i B} \frac{dT_{i0}}{dr}.}$$

- Given poloidal flow ($\Omega_{*\rm p} \equiv I\, U_{i\theta}/R^2$), relation of toroidal flow to $E_r$ is

$$\Omega_{\rm t} \equiv \vec{V} \cdot \vec{\nabla} \zeta = -\left( \frac{d\Phi}{d\psi_{\rm p}} + \frac{1}{n_i q_i} \frac{dp_i}{d\psi_{\rm p}} \right) + \Omega_{*\rm p} \quad \Longrightarrow \quad \boxed{V_{\rm t} \simeq \frac{E_r}{B_{\rm p}} - \frac{1}{n_i q_i B_{\rm p}} \frac{dp_i}{dr} + \frac{1.17}{q_i B_{\rm p}} \frac{dT_i}{dr}\,.}$$

# Low $\nu$ Flow Damping Can Be Included In $\parallel$ Viscous Stress

- A multi-collisionality parallel stress that yields the Braginskii and flux-surface-averaged (FSA) neoclassical closures has been proposed[4]

$$\pi_\parallel = \pi_\parallel^{\rm f} + \pi_\parallel^{\rm r},$$

$$\text{fast,} \quad \pi_\parallel^{\rm f} \equiv -\,3\,\eta_{00}\left(\frac{\vec{B}\cdot\vec{\nabla}\times(\vec{V}\times\vec{B})}{B^2} + \frac{2}{3}\vec{\nabla}\cdot\vec{V} - \frac{(\vec{B}\cdot\vec{V})(\vec{\nabla}\cdot\vec{B})}{B^2}\right),$$

$$\text{residual,} \quad \pi_\parallel^{\rm r} \equiv -\,mn\,\mu\,\langle B_0^2\rangle\frac{\hat{\rm b}\cdot\vec{\nabla}B_0}{\langle(\hat{\rm b}\cdot\vec{\nabla}B_0)^2\rangle}\,(U_\theta - U_\theta^0)\,, \quad \hat{\rm b} \equiv \vec{B}_0/B_0.$$

- Neoclassical poloidal flow damping frequency $\mu$ is of the form

$$\mu \simeq \frac{1.46\sqrt{\epsilon}\,\nu}{(1+\nu_*^{1/2}+\nu_*)(1+\epsilon^{3/2}\nu_*)}, \qquad \text{for collisionality parameter } \nu_* \equiv \frac{\nu}{\epsilon^{3/2}\omega_t} = \frac{R_0 q}{\epsilon^{3/2}\lambda}.$$

$\Longrightarrow$ banana regime for $\nu_* \ll 1$, plateau for $1 \ll \nu_* \ll \epsilon^{-3/2}$, Braginskii for $\nu_* \gg \epsilon^{-3/2}$.

- The "offset" poloidal flow velocity for ions is given by

$$U_{i\theta}^0(\psi_p) \simeq k_i\,\frac{I(\psi_p)}{q_i\langle B_0^2\rangle}\,\frac{dT_0}{d\psi_p}, \qquad \text{in which } k_i = \frac{\mu_{i01}/\mu_{i00}}{1+(\mu_{i11}-\mu_{i01}^2/\mu_{i00})/\nu_{i11}} \sim \frac{1.17}{1+0.67\sqrt{\epsilon}}.$$

# Plasma Resistivity Can Be Determined At Various Levels I

- Plasma resistivity can be estimated from electron force balance equation with $\vec{B} = \vec{0}$ assuming electron distribution is a flow-shifted Maxwellian:

$$0 = -\, n_e e \vec{E} + \vec{R}, \qquad \text{with} \;\; \vec{R} \equiv -\, m_e n_e \, \nu_e \, (\vec{V}_e - \vec{V}_i) = n_e e \, \eta_0 \, \vec{J}, \text{ collisional friction force,}$$

$$\Longrightarrow \qquad \vec{E} = \eta_0 \, \vec{J}, \qquad \text{in which} \qquad \boxed{\eta_0 \equiv \frac{m_e \nu_e}{n_e e^2}, \quad \text{reference } (\perp) \text{ electrical resistivity.}}$$

- But since $\nu \sim v^{-3}$ tail electrons suffers less collisions on average, the $\vec{E}$ field induces a heat flow and one must solve combination of flow & heat flow equations:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\, n_e e \vec{E} \\ 0 \end{bmatrix} + \begin{bmatrix} \vec{R}_J \\ \vec{R}_T \end{bmatrix}, \qquad \begin{bmatrix} \vec{R}_J \\ \vec{R}_T \end{bmatrix} \equiv -\, \frac{m_e n_e}{\tau_{ee}} \begin{bmatrix} Z & \frac{3}{2} Z \\ \frac{3}{2} Z & \sqrt{2} + \frac{13}{4} Z \end{bmatrix} \cdot \begin{bmatrix} \vec{V}_e - \vec{V}_i \\ -\, \frac{2}{5 n_e T_e} \vec{q}_e \end{bmatrix},$$

in which $\tau_{ee}$ is the reference electron collision time.

- For a plasma composed of electrons $(e)$, hydrogenic ions $(i,\, Z_i = 1)$ and impurity ions $(n_I,\, Z_I)$ with the same flow velocity as the hydrogenic ions

$$Z \to Z_{\text{eff}} \equiv \frac{n_i + \sum_I n_I Z_I^2}{n_e}, \qquad \text{which is typically} \sim 2\text{–}3 \text{ in tokamak plasmas.}$$

- Using the electron collisional friction coefficient matrix $\mathsf{L}_e$ (see p 22 in Lecture 2), the 2×2 matrix equation above can be written in the form (using $\nu_e \equiv Z_{\text{eff}} / \tau_{ee}$)

$$\begin{bmatrix} \vec{R}_J \\ \vec{R}_T \end{bmatrix} = \frac{n_e e \, \eta_0}{Z_{\text{eff}}} \mathsf{L}_e \cdot \begin{bmatrix} \vec{J} \\ \frac{2e}{5T_e} \vec{q}_e \end{bmatrix} \equiv \begin{bmatrix} n_e e \vec{E} \\ 0 \end{bmatrix}, \qquad \mathsf{L}_e \equiv \begin{bmatrix} Z_{\text{eff}} & \frac{3}{2} Z_{\text{eff}} \\ \frac{3}{2} Z_{\text{eff}} & \sqrt{2} + \frac{13}{4} Z_{\text{eff}} \end{bmatrix}.$$

# Plasma Resistivity Determination II: Spitzer Conductivity

- Current and heat flow are obtained by inverting the friction matrix $\mathsf{L}_e$:

$$\begin{bmatrix} \vec{J} \\ \frac{2e}{5T_e}\vec{q}_e \end{bmatrix} = \frac{Z_{\text{eff}}}{\eta_0}\,[\mathsf{L}_e]^{-1}\cdot\begin{bmatrix} \vec{E} \\ 0 \end{bmatrix},$$ whose first row yields the Ohm's law $\vec{J} = \sigma^{\text{Sp}}\vec{E}$.

- The Spitzer electrical conductivity $\sigma^{\text{Sp}}$ includes electron heat flow effects via their effects in the inverse of the $2\times 2$ matrix $\mathsf{L}_e$ of friction coefficients:

$$\boxed{\sigma^{\text{Sp}} \equiv \frac{Z_{\text{eff}}}{\eta_0}\,[\mathsf{L}_e]^{-1}_{00} = \frac{1}{\eta_0}\,\frac{\sqrt{2}+(13/4)Z_{\text{eff}}}{\sqrt{2}+Z_{\text{eff}}}} \quad\Longrightarrow\quad \frac{1.93}{\eta_0}\ \text{for}\ Z_{\text{eff}}=1,\quad \frac{3.25}{\eta_0}\ \text{for}\ Z_{\text{eff}}\to\infty.$$

The standard Spitzer/Braginskii coefficient for $Z = 1$ is 1.96, which differs from this $2\times 2$ matrix result by less than 2%. Greater accuracy is obtained from $3\times 3$ or higher order matrices that take account of energy-weighted heat flow etc., but is unwarranted because the collision operator is only accurate to $\mathcal{O}\{1/\ln\Lambda\} \sim 1/17 \sim 6\%$.

- In a uniform magnetic field $\vec{B}$ the plasma electrical conductivity is anisotropic:

$\sigma_\parallel \equiv \sigma_\parallel^{\text{Sp}} > 1/\eta_0,$ because $\vec{E}_\parallel$ causes electron heat flow along $\vec{B}$, but

$\sigma_\perp \equiv 1/\eta_0,$ because $\vec{E}_\perp$ causes heat flow in $\vec{E}_\perp\times\vec{B}$, not $\vec{E}_\perp$ direction.

# Plasma Resistivity Determination III: Viscosity Effects

- The magnetic field varies poloidally in tokamaks. It induces parallel viscous and viscous heat forces that add to FSA parallel force balance equations:

$$\begin{aligned} 0 &= -n_e e\langle \vec{B}_0 \cdot \vec{E}^A \rangle + \langle \vec{B}_0 \cdot \vec{R}_J \rangle - \langle \vec{B}_0 \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_{e\|} \rangle, \qquad && \langle \vec{B}_0 \cdot \vec{\nabla}\phi \rangle = 0,\ \langle \vec{B}_0 \cdot \vec{\nabla} p_e \rangle = 0, \\ 0 &= \langle \vec{B}_0 \cdot \vec{R}_T \rangle - \langle \vec{B}_0 \cdot \vec{\nabla} \cdot \overleftrightarrow{\Theta}_{e\|} \rangle, && \langle \vec{B}_0 \cdot \vec{\nabla} T_e \rangle = 0. \end{aligned}$$

- In matrix form the FSA parallel viscous flow and heat flow forces are:

$$\begin{bmatrix} \langle \vec{B}_0 \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_{e\|} \rangle \\ \langle \vec{B}_0 \cdot \vec{\nabla} \cdot \overleftrightarrow{\Theta}_{e\|} \rangle \end{bmatrix} \equiv \frac{m_e n_e}{\tau_{ee}} \mathsf{M}_e \cdot \begin{bmatrix} \langle B_0^2 \rangle U_{e\theta} \\ \langle B_0^2 \rangle Q_{e\theta} \end{bmatrix} = \frac{m_e n_e}{\tau_{ee}} \mathsf{M}_e \cdot \begin{bmatrix} -\frac{1}{n_e e}(\langle \vec{B}_0 \cdot \vec{J} \rangle + I \frac{dP}{d\psi_p}) + U_{i\theta} \\ \frac{-2}{n_e T_e} \langle \vec{B}_0 \cdot \vec{q}_e \rangle + \frac{I}{e} \frac{dT_e}{d\psi_p} \end{bmatrix},$$

$$\mathsf{M}_e \equiv \begin{bmatrix} \mu_{e00} & \mu_{e01} \\ \mu_{e01} & \mu_{e11} \end{bmatrix} \sim 1.46\sqrt{\epsilon} \begin{bmatrix} 0.533 + Z_{\rm eff} & 0.625 + \frac{3}{2} Z_{\rm eff} \\ 0.625 + \frac{3}{2} Z_{\rm eff} & 1.386 + \frac{13}{4} Z_{\rm eff} \end{bmatrix}.$$

- Matrix equation for the FSA parallel force, heat force balances becomes

$$\frac{\eta_0}{Z_{\rm eff}} \left( [\mathsf{L}_e + \mathsf{M}_e] \cdot \begin{bmatrix} \langle \vec{B}_0 \cdot \vec{J} \rangle \\ \frac{2e}{5T_e} \langle \vec{B}_0 \cdot \vec{q}_e \rangle \end{bmatrix} + \mathsf{M}_e \cdot \begin{bmatrix} I\, dP/d\psi_p - n_e e \langle B_0^2 \rangle U_{i\theta} \\ -n_e I\, dT_e/d\psi_p \end{bmatrix} \right) = \begin{bmatrix} \langle \vec{B}_0 \cdot \vec{E}^A \rangle \\ 0 \end{bmatrix}.$$

- This is solved for parallel current $\langle \vec{B}_0 \cdot \vec{J} \rangle$ by inverting $[\mathsf{L}_e + \mathsf{M}_e]$ matrix.

# Resistivity Determination IV: Neoclassical Ohm's Law

- Neoclassical FSA parallel Ohm's law that results from matrix inversion is[4]

$$\boxed{\langle \vec{B}_0 \cdot \vec{E}^A \rangle = \eta_\parallel^{\rm nc} \, ( \, \langle \vec{B}_0 \cdot \vec{J} \rangle - \langle \vec{B}_0 \cdot \vec{J}_{\rm bs} \rangle \, ),} \quad \text{neoclassical parallel Ohm's law.}$$

- The parallel neoclassical resistivity $\eta_\parallel^{\rm nc}$ is

$$\boxed{\eta_\parallel^{\rm nc} \equiv \frac{\eta_0/Z_{\rm eff}}{[\mathsf{L}_e + \mathsf{M}_e]_{00}^{-1}} \sim \eta_0 \left( 1 + \frac{\mu}{\nu} \right)} \quad \overset{|\mathsf{L}_e| \gg |\mathsf{M}_e|}{\Longrightarrow} \quad \frac{\eta_0/Z_{\rm eff}}{[\mathsf{L}_e]_{00}^{-1}} \equiv \frac{1}{\sigma_\parallel^{\rm Sp}} \, .$$

- The bootstrap current $\langle \vec{B}_0 \cdot \vec{J}_{\rm bs} \rangle$ is driven by radial pressure gradient $dP/d\psi_{\rm p}$:

$$\langle \vec{B}_0 \cdot \vec{J}_{\rm bs} \rangle = \big[ [\mathsf{L}_e + \mathsf{M}_e]^{-1} \cdot \mathsf{M}_e \big]_{00} \left( -I \frac{dP}{d\psi_{\rm p}} + n_e e \langle B_0^2 \rangle U_{i\theta} \right) + \big[ [\mathsf{L}_e + \mathsf{M}_e]^{-1} \cdot \mathsf{M}_e \big]_{01} \left( n_e I \frac{dT_e}{d\psi_{\rm p}} \right)$$

$$\sim - \frac{\mu}{\nu + \mu} \, I \, \frac{dP}{d\psi_{\rm p}} \sim - \sqrt{\epsilon} \, \frac{B_0}{B_p} \frac{dP}{dr}.$$

- In order to extend this to a local description, note that bootstrap drive is

$$I \frac{dP}{d\psi_p} \equiv B_0^2 \frac{\vec{J}_\perp \cdot \vec{\nabla} \theta}{\vec{B}_0 \cdot \vec{\nabla} \theta} \quad \Longrightarrow \quad B^2 \frac{\vec{J}_\perp \cdot \vec{\nabla} \theta}{\vec{B} \cdot \vec{\nabla} \theta}, \quad \text{for } \vec{J}_\perp \equiv \frac{\vec{B} \times \vec{\nabla} P}{B^2} \text{ with } \vec{B} \cdot \vec{\nabla} P = 0 \Longrightarrow P(\psi_{\rm p}).$$

# Tokamak Extended MHD Model Is Obtained From Fluid Equations

- Plasma density and charge continuity equations result from sums over species:

$$\sum_s n_s m_s \quad \Longrightarrow \quad \boxed{\left.\frac{\partial \rho_m}{\partial t}\right|_{\vec{x}} + \vec{\nabla}\cdot\rho_m\vec{V} = \sum_s m_s S_{ns},} \qquad \textstyle\sum_s n_s q_s \quad \Longrightarrow \quad \boxed{\vec{\nabla}\cdot\vec{J} = 0.}$$

- Total plasma equation of state (entropy eqn.) is unchanged from usual form (p 3).

- Plasma force balance is obtained by summing momentum equations over species:[4]

$$\boxed{\left.\frac{\partial(\rho_m\vec{V})}{\partial t}\right|_{\vec{x}} + \vec{\nabla}\cdot(\rho_m\vec{V}\vec{V}) = \vec{J}\times\vec{B} - \vec{\nabla}P - \sum_s(\vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}{}^{\rm f}_{s\|} + \vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}{}^{\rm r}_{s\|} + \vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}_{s\wedge}) + \sum_s \vec{S}_{\rm ps}.}$$

- General Ohm's law is obtained from electron force balance equation ($\hat{\rm b} \equiv \vec{B}/B$):

$$\vec{E} = -\vec{V}\times\vec{B} + \frac{\vec{J}\times\vec{B} - \vec{\nabla}p_e - \vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}{}^{\rm f}_{e\|} - \vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}_{e\wedge} - C_{\nabla T} n_e\,\hat{\rm b}\,(\hat{\rm b}\cdot\vec{\nabla}T_e) + \vec{S}_{\rm pe}}{n_e e}$$
$$+\frac{1}{\sigma_\perp}\left(\vec{J}_\perp - \frac{3}{2}\frac{n_e\vec{B}\times\vec{\nabla}T_e}{B^2}\right) + \eta^{\rm nc}_\|\left(\vec{J}_\| - \vec{J}_{\|\rm drives}\right) - \frac{m_e}{e}\left(\frac{\partial}{\partial t} + \vec{V}_e\cdot\vec{\nabla}\right)\vec{V}_e,$$

$$\vec{J}_\perp \equiv -\hat{\rm b}\times(\hat{\rm b}\times\vec{J}), \qquad \vec{J}_\| \equiv \hat{\rm b}\,(\hat{\rm b}\cdot\vec{J}).$$

- Use $\vec{J}_{\|\rm drives} \equiv \dfrac{\vec{B}}{B}\langle\vec{B}_0\cdot\vec{J}_{\rm drives}\rangle$ fom p 32, but $I\,\dfrac{dP}{d\psi_p} \to B^2\dfrac{\vec{J}_\perp\cdot\vec{\nabla}\theta}{\vec{B}\cdot\vec{\nabla}\theta}$ in $\langle\vec{B}_0\cdot\vec{J}_{\rm bs}\rangle$.

# Key Properties Of Tokamak Extended MHD Model

- **Tokamak extended MHD model adds collisional effects for $t > 1/\nu_s$**

  **primarily via the viscous forces due to the parallel viscous stresses $\overleftrightarrow{\pi}_{s\|}$, which**

  **for $t > 1/\nu_e \simeq 0.2$ ms modifies parallel Ohm's law by increasing $\|$ resistivity plus adds bootstrap current driven by the radial plasma pressure gradient,**

  **and for $t > 1/\nu_i \simeq 34$ ms damps the poloidal ion flow to $U_{i\theta} \propto dT_i/d\psi_\mathrm{p}$ and increases the plasma's $\perp$ flow inertia from $\propto \rho_m/B^2$ to $\propto \rho_m/B_\mathrm{p}^2$.**

- **It is important to recall that the extended MHD model "owns" the current density $\vec{J}$ because**

  **in MHD models $\vec{J} = \vec{\nabla}\times\vec{B}/\mu_0$ in which the magnetic field is determined from Faraday's law $\partial\vec{B}/\partial t = -\vec{\nabla}\times\vec{E}$ with the electric field being determined from the extended MHD Ohm's law (preceding viewgraph),**

  **and proper solutions of the Chapman-Enskog kinetic equation yield kinetic distortions $F_s$ that have no momentum moments (i.e., $\int d^3v\, m_s\vec{v}\, F_s = \vec{0}$) and hence produce no contributions to $\vec{J}$.**

- **Next (final) step will be to obtain net radial transport equations for a tokamak plasma on the long transport time scale $t \gg 1/\nu_s$.**

## Next Step: Develop Modern Transport Equations For Tokamaks

- **Tokamak plasma transport equations for modeling codes (e.g., ONETWO, TRANSP) were developed in late 70's from $n$, $T$ fluid moment equations with collisional Braginskii closures; and then *ad hoc* terms are added for**

  **neoclassical effects on $\|$ Ohm's law (trapped particle effects on $\eta_\|$ and bootstrap current),**

  **fluctuation-induced transport induced by micro-turbulence,**

  **heating & current-drive and flow sources & sinks,**

  **effects of small 3-D magnetic field asymmetries, etc.**

- **But tokamak plasmas are not in a collisional regime! ($\lambda \gg Rq$) — and transport equations that naturally include all these effects should be developed.**

- **Here, self-consistent fluid-moment-based radial transport equations that include all these effects for nearly axisymmetric single-ion-species tokamak plasmas will be developed[6,7] using neoclassical-based closures.**

- **The procedures used (solve for flows in flux surfaces first) and net plasma transport equations are analogous to those developed for stellarator transport.[8]**

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[6] J.D. Callen, A.J. Cole and C.C. Hegna, "Toroidal rotation in tokamak plasmas," Nucl. Fusion **49**, 085021 (2009).

[7] J.D. Callen, A.J. Cole, and C.C. Hegna, "Toroidal flow and particle flux in tokamak plasmas," Phys. Plasmas **16**, 082504 (2009); Erratum Phys. Plasmas **20**, 069901 (2013).

[8] See for example K.C. Shaing and J.D. Callen, Phys. Fluids **26**, 3315 (1983) and references cited therein.

# Velocity Moments Of PKE Yield Fluid Moment Equations

- **Start from the conservative form of the 6-D plasma kinetic equation (PKE) that includes the Fokker-Planck Coulomb collision operator $\mathcal{C}\{f\}$ and a kinetic sources operator $\mathcal{S}\{f\}$, in laboratory coordinates ($\vec{x}$):**

$$\left.\frac{\partial f}{\partial t}\right|_{\vec{x}} + \frac{\partial}{\partial \vec{x}} \cdot [\vec{v}\, f] + \frac{\partial}{\partial \vec{v}} \cdot \left[\frac{q}{m}(\vec{E} + \vec{v}\times\vec{B})\, f\right] = \mathcal{C}\{f\} + \mathcal{S}\{f\}.$$

- **Take velocity-space moments [ $\int d^3v\,(1, m\vec{v}, mv^2/2)$ ] of this PKE to obtain fluid moment equations for each species in their conservative forms ($p \equiv nT$):**

density $$\left.\frac{\partial n}{\partial t}\right|_{\vec{x}} + \vec{\nabla}\cdot n\vec{V} = S_n,$$

momentum $$\left.\frac{\partial}{\partial t}\right|_{\vec{x}}(mn\vec{V}) + \vec{\nabla}\cdot(mn\,\vec{V}\vec{V}) = nq\,(\vec{E} + \vec{V}\times\vec{B}) - \vec{\nabla}p - \vec{\nabla}\cdot\overleftrightarrow{\pi} + \vec{R} + \vec{S}_{\rm p},$$

energy $$\frac{3}{2}\left.\frac{\partial p}{\partial t}\right|_{\vec{x}} + \vec{\nabla}\cdot\left(\vec{q} + \frac{5}{2}\,p\,\vec{V}\right) = Q + \vec{V}\cdot\vec{\nabla}p - \overleftrightarrow{\pi}:\vec{\nabla}\vec{V} + S_{\varepsilon}.$$

- **Determine closures for $\overleftrightarrow{\pi}$, $\vec{q}$ kinetically from CEKE — do not use Braginskii.**

- **Luckily, only FSA parallel viscous forces $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_{s\|}\rangle$ will be needed.**

# A Number Of Assumptions Facilitate The Analysis

**1)** *Possible extended MHD macroinstabilities are stabilized or controlled.*

**2)** *Small gyroradius expansion*, **which to zeroth order yields magnetohydrodynamic (MHD) radial force balance equilibrium, flows within flux surfaces at first order, and second order transport fluxes across flux surfaces.**

**3)** *Axisymmetric nested flux surfaces* **to lowest order (i.e., no magnetic islands in region of interest).**

**4)** *Gyroradius-small $\vec{B}$ non-axisymmetries* **(NA), by assuming 2-D toroidal axisymmetry to lowest order and that 3-D toroidal non-axisymmetries in the magnetic field $\vec{B}$ are first order in the gyroradius expansion.**

**5)** *Banana-plateau collisionalty regime* **where collision lengths are long compared to plasma toroidal circumference so plasma properties are constant on magnetic flux surfaces — valid almost out to the 2-D divertor separatrix.**

**6)** *Gyroradius small plasma fluctuations* **which lead mostly to second order "anomalous" plasma transport across flux surfaces.**

**7)** *Slow poloidal magnetic field transients and weak sources and sinks* **that occur and contribute on the plasma transport or longer time scale.**

# Multi-Stage Strategy Is Used To Develop Transport Equations[7,9]

- **I.** **Average the density, momentum and energy equations over fluctuations (i.e., average over toroidal angle $\zeta$) and then flux-surface-average (FSA) them.**

- **II.** *Key Elements Of New Approach:* **Consider sequentially specific components of the equilibrium force balance equations and their consequences:**

  **IIA.** *Radial:* **Use zeroth order radial force balance enforced by compressional Alfvén waves to obtain relation between toroidal & poloidal flows and radial electric field $E_\rho$ & $dp_i/d\rho$.**

  **IIB.** *Parallel (poloidal):* **Determine parallel neoclassical Ohm's law and first order poloidal flows & heat flows within a flux surface from equilibrium momentum & heat flux equations.**

  **IIC.** *Toroidal:* **Require net radial current from all particle fluxes to vanish and thereby determine FSA toroidal momentum equation, and hence toroidal rotation $\Omega_\mathrm{t}$ (and thus $E_\rho$).**

- **III.** **Substitute net second order ambipolar fluxes into FSA transport equations to obtain final comprehensive "radial" transport equations — for ambipolar $n_e = Z_i n_i$, $p_e$, $p_i$, and $\Omega_\mathrm{t} \equiv \vec{V}\cdot\vec{\nabla}\zeta \simeq V_\mathrm{t}/R$ (toroidal plasma rotation frequency).**

[9] J.D. Callen, C.C. Hegna, and A.J. Cole, "Transport equations in tokamak plasmas," Phys. Plasmas **17**, 056113 (2010).

# Natural $\vec{B}$-Field-Based Tokamak Coordinates Are Non-Orthogonal

- *Coordinates* $\boldsymbol{\rho, \theta, \zeta}$. **Toroidal-flux-surface-based radial variable is defined by $\rho \equiv \sqrt{\psi_t/\pi B_{t0}}$ (m) plus poloidal ($\theta$), toroidal ($\zeta$) angles are used, non-orthogonal:**

$$\vec{e}^{\,\rho} \equiv \vec{\nabla}\rho,\ \vec{e}^{\,\theta} = \vec{\nabla}\theta,\ \vec{e}^{\,\zeta} = \vec{\nabla}\zeta, \quad \vec{e}_\rho = \sqrt{g}\,\vec{\nabla}\theta\times\vec{\nabla}\zeta,\ \vec{e}_\theta = \sqrt{g}\,\vec{\nabla}\zeta\times\vec{\nabla}\rho,\ \vec{e}_\zeta = \sqrt{g}\,\vec{\nabla}\rho\times\vec{\nabla}\theta,$$

$$\sqrt{g} \equiv 1/\vec{\nabla}\rho\cdot\vec{\nabla}\theta\times\vec{\nabla}\zeta = \psi_{\rm p}' q R^2/I, \quad \text{and from axisymmetry } \vec{e}_\zeta = R^2\vec{\nabla}\zeta = R\,\hat{\vec{e}}_\zeta,\ \hat{\vec{e}}_\zeta \equiv \vec{\nabla}\zeta/|\vec{\nabla}\zeta|.$$

- *Average* $\vec{B}_0$. **Lowest order axisymmetric equilibrium $\vec{B}_0$ is represented in terms of the poloidal magnetic flux $\psi_{\rm p}(\rho, t)$:**

$$\vec{B}_0(\rho,\theta) \equiv \vec{B}_{\rm t} + \vec{B}_{\rm p} \equiv I\vec{\nabla}\zeta + \vec{\nabla}\zeta\times\vec{\nabla}\psi_{\rm p} = \vec{\nabla}\psi_{\rm p}\times\vec{\nabla}(q\theta - \zeta), \qquad I(\psi_{\rm p}) = RB_t.$$

- $\|$, $\perp$ *Directions.* **Parallel, perpendicular directions are relative to $\vec{B}_0$:**

$$\vec{A}_\| \equiv (\vec{B}_0\cdot\vec{A})/B_0, \qquad \vec{A}_\perp \equiv -\vec{B}_0\times(\vec{B}_0\times\vec{A})/B_0^2.$$

- *Flux-Surface-Averaging (FSA).* **Has key properties $[V(\rho) \equiv \int_0^\rho d^3x = \text{volume}]$:**

$$\langle f(\vec{x})\rangle \equiv \frac{\int d\theta \int d\zeta \sqrt{g}\, f(\vec{x})}{\int d\theta \int d\zeta \sqrt{g}}, \quad \langle \vec{B}_0\cdot\vec{\nabla}f\rangle = 0, \quad \langle \vec{\nabla}\cdot\vec{A}\rangle = \frac{d}{dV}\langle \vec{A}\cdot\vec{\nabla}V\rangle = \frac{1}{V'}\frac{d}{d\rho}(V'\langle\vec{A}\cdot\vec{\nabla}\rho\rangle).$$

# A Small Gyroradius Expansion Is Used

- *Gyroradius Expansion.* **Order terms and physical processes such as equilibrium, Pfirsch-Schlüter flows, non-axisymmetries (NA) and fluctuations as**

$$\varrho_* \sim \varrho_i/a \ll 1, \quad p(\vec{x}) = \underbrace{p_0(\rho)}_{\text{equil.}} + \varrho_* [\underbrace{\bar{p}_1(\rho,\theta)}_{\text{PS var.}} + \underbrace{\tilde{p}_1(\rho,\theta,\zeta)}_{\text{NA + fluct.}}] + \mathcal{O}\{\varrho_*^2\}, \quad \overline{\tilde{p}_1} = 0.$$

- *Fourier Expansion.* **Due to toroidal symmetry, Fourier expand $\zeta$ dependence:**

$$\tilde{p}_1 = \textstyle\sum_n \hat{p}_n e^{-in\zeta}, \quad \hat{p}_n \equiv \frac{1}{2\pi}\int_0^{2\pi} d\zeta\, e^{in\zeta}\,\tilde{p}_1, \quad \overline{p(\vec{x})} \equiv \frac{1}{2\pi}\int_0^{2\pi} d\zeta\, p(\vec{x}) = p_0(\rho) + \varrho_* \bar{p}_1(\rho,\theta) + \mathcal{O}\{\varrho_*^2\}.$$

- *Fluctuation Derivatives.* **Large perpendicular derivatives of fluctuations:**

$\nabla_\perp \tilde{p}_1 \sim (1/\varrho_*)\, \varrho_* \sim \varrho_*^0$, **but** $\nabla_\parallel \tilde{p}_1 \sim \varrho_*^0 \varrho_* \sim \varrho_*$; $\quad \perp$ **derivatives of** $p_0$, $\bar{p}_1$ **will be** $\mathcal{O}\{\varrho_*^0, \varrho_*\}$.

- *Magnetic Field.* **Represent as average** $\vec{B}_0 \equiv \vec{\nabla}\times(\psi_{\rm t}\vec{\nabla}\theta - \psi_{\rm p}\vec{\nabla}\zeta)$ **plus small** $\mathcal{O}\{\varrho_*\}$ **perturbations** $\tilde{\vec{B}}$ **due to 3-D NA and collective plasma fluctuations:**

$$\vec{B} = \vec{B}_0(\rho,\theta) + \varrho_*\,(\tilde{\vec{B}}_\parallel + \tilde{\vec{B}}_\perp) + \mathcal{O}\{\varrho_*^2\}, \qquad |\vec{B}| \simeq B_0(\rho,\theta) + \varrho_*\,\tilde{B}_\parallel + \mathcal{O}\{\varrho_*^2\}.$$

- *Electric Field.* **Represent as a sum of scalar and vector potentials:**

$$\vec{E} = -\vec{\nabla}\phi + \vec{E}^A, \qquad \vec{E}^A \equiv -\frac{\partial \vec{A}}{\partial t}, \qquad \bar{\vec{E}}^A = \left(\frac{\partial \Psi}{\partial t} + \dot{\psi}_{\rm p}\right)\vec{\nabla}\zeta - \dot{\psi}_{\rm t}\vec{\nabla}\theta \sim \mathcal{O}\{\varrho_*^2\}.$$

## I. Average Moment Equations Over Fluctuations, Then FSA[6,7,9]

- First, use perturbation procedure outlined on preceding viewgraph.

- Next, $\zeta$-average over fluctuations (overbar) and FSA ($\langle\cdots\rangle$) density, energy equations [ $V(\rho) \equiv \int_0^\rho d^3x$, $V' \equiv dV(\rho)/d\rho$, $\rho \equiv \sqrt{\psi_t/\pi B_0}$ ] for each species $s$:

$$\text{density} \quad \left.\frac{\partial n_0}{\partial t}\right|_{\vec{x}} + \frac{1}{V'}\frac{\partial}{\partial \rho}(V'\,\Gamma) = \langle \bar{S}_n \rangle, \qquad \boxed{\Gamma \equiv \langle\,(n_0\bar{\vec{V}}_2 + \overline{\tilde{n}_1\tilde{\vec{V}}_1})\cdot\vec{\nabla}\rho\rangle,}$$

$$\text{energy} \quad \frac{3}{2}\left.\frac{\partial p_0}{\partial t}\right|_{\vec{x}} + \frac{1}{V'}\frac{\partial}{\partial \rho}\left[V'\left\langle\left(\bar{\vec{q}}_2 + \frac{5}{2}\,(\,p_0\vec{V}_2 + \overline{\tilde{p}_1\tilde{\vec{V}}_1}\,)\right)\cdot\vec{\nabla}\rho\right\rangle\right]$$

$$= \langle \bar{Q}_\Delta \rangle - \left\langle \bar{\vec{R}}_1\cdot\bar{\vec{V}}_1 + \overline{\tilde{\vec{R}}_1\cdot\tilde{\vec{V}}_1}\right\rangle + \left\langle \bar{\vec{V}}_2\cdot\vec{\nabla}p_0 + \overline{\tilde{\vec{V}}_1\cdot\vec{\nabla}\tilde{p}_1}\right\rangle - \left\langle \overleftrightarrow{\pi} : \vec{\nabla}\bar{\vec{V}}_1 \right\rangle + \langle \bar{S}_\varepsilon \rangle.$$

- Finally, similarly average the momentum (force balance) equation and determine its radial ($\vec{\nabla}\rho\,\cdot$) component and the flux surface average (FSA) of its parallel ($\vec{B}_0\,\cdot$) and toroidal angular ($\vec{e}_\zeta\,\cdot = R\,\hat{\vec{e}}_\zeta\,\cdot$) components:

$$\text{radial } \mathcal{O}\{\varrho_*^0\} \quad mn_0\frac{\partial\vec{V}}{\partial t} = nq(\vec{E}+\vec{V}\times\vec{B}) - \vec{\nabla}p - \vec{\nabla}\cdot\overleftrightarrow{\pi} \quad \stackrel{\sum_s}{\Longrightarrow} \quad \rho_m\frac{\partial\vec{V}}{\partial t} = \vec{J}\times\vec{B} - \vec{\nabla}P - \vec{\nabla}\cdot\overleftrightarrow{\Pi},$$

$$\text{parallel } \mathcal{O}\{\varrho_*\} \quad mn_0\frac{\partial\langle\vec{B}_0\cdot\bar{\vec{V}}\rangle}{\partial t} = n_0q\langle\vec{B}_0\cdot\bar{\vec{E}}^A\rangle - \langle\vec{B}_0\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}\rangle + \langle\vec{B}_0\cdot\bar{\vec{R}}\rangle - mn_0\langle\vec{B}_0\cdot\overline{\tilde{\vec{V}}\cdot\vec{\nabla}\tilde{\vec{V}}}\,\rangle + \cdots,$$

$$\text{toroidal } \mathcal{O}\{\varrho_*^2\} \quad \left.\frac{\partial}{\partial t}\right|_{\vec{x}}\langle\vec{e}_\zeta\cdot mn_0\bar{\vec{V}}\rangle = \boxed{q\,\Gamma} - \langle\vec{e}_\zeta\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}\rangle - \langle\vec{\nabla}\cdot mn\,\overline{(\vec{e}_\zeta\cdot\tilde{\vec{V}})\tilde{\vec{V}}}\rangle + \cdots.$$

## II. Order $\varrho_*^0, \varrho_*^1, \varrho_*^2$ Force Balances And Flows Are Different

- $\varrho_*^0$. Zeroth order fluid moment equations yield ideal MHD model.

- IIA. Compressional Alfvén waves $\perp$ to $\vec{B}_0$ enforce $\vec{J}_0\times\vec{B}_0 = \vec{\nabla}P_0$ plus Ohm's law $\vec{E}_0+\vec{V}\times\vec{B}_0 = (\vec{J}_0\times\vec{B}_0-\vec{\nabla}p_e)/n_e e = -\vec{\nabla}p_i/n_e e$ yields radial force balance:

$$0 = \vec{e}_\rho\cdot[n_i q_i(\vec{E}+\vec{V}\times\vec{B}) - \vec{\nabla}p_i] \quad\Longrightarrow\quad \Omega_{\rm t}\equiv\vec{V}\cdot\vec{\nabla}\zeta = -\left(\frac{d\Phi}{d\psi_{\rm p}} + \frac{1}{n_i q_i}\frac{dp_i}{d\psi_{\rm p}} - q\,\vec{V}\cdot\vec{\nabla}\theta\right)$$

$$\Longrightarrow\quad \boxed{V_{\rm t}\simeq\frac{E_\rho}{B_{\rm p}} - \frac{1}{n_i q_i}\frac{dp_i}{d\rho} + \frac{B_{\rm t}}{B_{\rm p}}V_{\rm p}\,,}$$

relation between toroidal, poloidal flows and $E_\rho$, $dp_i/d\rho$.

- Maxwellianization of electron, ion distributions on their collision times of $1/\nu_e, 1/\nu_i$ cause $n$, $T$ to be constant over collision lengths $\lambda_e, \lambda_i$ and hence on flux surfaces, and flows $\vec{V}$ become physically meaningful.

- $\varrho_*^1$. First order flows are on magnetic flux surfaces ($\theta, \zeta$ or $\wedge, \|$ directions):

$$\vec{V}_1\equiv\underbrace{\vec{e}_\theta\,(\vec{V}\cdot\vec{\nabla}\theta)}_{\text{poloidal}}+\underbrace{\vec{e}_\zeta\,(\vec{V}\cdot\vec{\nabla}\zeta)}_{\text{toroidal}}=\underbrace{V_\|\vec{B}_0/B_0}_{\text{parallel}}+\underbrace{\vec{V}_\wedge}_{\text{cross}},\qquad \vec{V}_{s\wedge}\equiv\underbrace{\frac{\vec{B}_0\times\vec{\nabla}\psi_p}{B_0^2}\left(\frac{d\phi}{d\psi_{\rm p}}+\frac{1}{n_{s0}q_s}\frac{dp_s}{d\psi_{\rm p}}\right)}_{\vec{E}\times\vec{B}\text{ and diamagnetic}}.$$

- $\varrho_*^2$. Radial flows $\perp$ to flux surfaces are second order: $\vec{V}_2\cdot\vec{\nabla}\psi_{\rm p}\neq 0$ — to calculate, need to determine flows in surface first, as in stellarators.[8]

## IIB. Electron Parallel Force Balance Yields FSA Parallel Ohm's Law

- For times $t > 1/\nu_e \sim 0.2$ ms, FSA of equilibrium parallel force balance becomes

$$0 = -n_e e\langle \vec{B}\cdot\vec{E}^A\rangle - \langle \vec{B}\cdot\vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}_e\rangle + \langle \vec{B}\cdot\vec{R}_e\rangle + \langle \vec{B}\cdot\vec{S}_{\rm pe}\rangle - m_e n_{e0}\langle \vec{B}\cdot\overline{\tilde{\vec{V}}_e\cdot\vec{\nabla}\tilde{\vec{V}}_e}\,\rangle - n_{e0}e\langle \vec{B}\cdot\overline{\tilde{\vec{V}}_e\times\tilde{\vec{B}}_\perp}\rangle.$$

- Using the collisional friction relation $\vec{B}_0\cdot\bar{\vec{R}}_e = -\bar{\vec{B}}_0\cdot\vec{R}_i \simeq n_{e0}\,e\,\vec{B}_0\cdot\vec{J}/\sigma_\parallel$ and neoclassical closure $\langle \vec{B}_0\cdot\vec{\nabla}\cdot\bar{\overset{\leftrightarrow}{\pi}}_{e\parallel}\rangle \simeq m_e n_{e0}\langle B_0^2\rangle(\mu_{e00}U_{e\theta} + \mu_{e01}Q_{e\theta})$, this equation yields an extended neoclassical-based parallel Ohm's law:[10]

$$\underbrace{\langle \vec{B}_0\cdot\bar{\vec{E}}^A\rangle}_{\bar{E}_\parallel^A\ \text{field}} = \eta_\parallel^{\rm nc}\underbrace{\langle \vec{B}_0\cdot\vec{J}\rangle}_{\parallel\ \text{current}} - \frac{1}{\sigma_\parallel}\Big[\underbrace{\langle \vec{B}_0\cdot\vec{J}_{\rm bs}\rangle}_{\text{bootstrap}} + \underbrace{\langle \vec{B}_0\cdot\vec{J}_{\rm CD}\rangle}_{\text{current drive}} + \underbrace{\langle \vec{B}_0\cdot\vec{J}_{\rm dyn}\rangle}_{\text{dynamo}}\Big], \qquad \eta_\parallel^{\rm nc} \simeq \frac{1}{\sigma_\parallel}\left(1 + \frac{\sigma_\parallel}{\sigma_\perp}\frac{\mu_{e00}}{\nu_e}\right).$$

- $\parallel$ currents are driven by $dP_0/d\psi_{\rm p}$, $\parallel$ $e$ momentum sources and fluctuations:

$$\langle \vec{B}_0\cdot\vec{J}_{\rm bs}\rangle \simeq -\frac{\sigma_\parallel}{\sigma_\perp}\frac{\mu_{e00}}{\nu_e}\left(I\frac{dP_0}{d\psi_p} - n_{e0}eU_{i\theta}\langle B_0^2\rangle\right), \qquad \text{bootstrap current,}$$

$$\langle \vec{B}_0\cdot\vec{J}_{\rm CD}\rangle \equiv -\frac{\sigma_\parallel}{n_{e0}e}\langle \vec{B}_0\cdot\left(\bar{\vec{S}}_{\rm pe} - m_e\bar{\vec{V}}_e\bar{S}_{ne}\right)\rangle, \qquad \text{non-inductive current drive,}$$

$$\langle \vec{B}_0\cdot\vec{J}_{\rm dyn}\rangle = \underbrace{\frac{\sigma_\parallel}{n_{e0}e}\langle \vec{B}_0\cdot\left(m_e n_{e0}\overline{\tilde{\vec{V}}_e\cdot\vec{\nabla}\tilde{\vec{V}}_e} + \overline{\vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}_{e\wedge}}\right)\rangle}_{\parallel\ \text{Reynolds stress}} + \underbrace{\sigma_\parallel\langle \vec{B}_0\cdot\overline{\tilde{\vec{V}}_e\times\tilde{\vec{B}}_\perp}\,\rangle}_{\parallel\ \text{Maxwell stress}}, \qquad \text{dynamo.}$$

[10] For illustrative purposes the equations here are simplified versions where the effects of the poloidal electron heat flow $Q_{e\theta}$ have been neglected.

# IIB. Parallel Current Properties Can Also Be Obtained

- Adding first order flows, one obtains the usual sum of the parallel and diamagnetic current densities:
$$\vec{J} \equiv \sum_s n_{s0}\, q_s \vec{\bar{V}}_{s1} \equiv \vec{J}_\parallel + \vec{J}_\wedge = J_\parallel \frac{\vec{B}_0}{B_0} + \frac{\vec{B}_0 \times \vec{\nabla} P_0(\psi_\mathrm{p})}{B_0^2}.$$

- Summing the poloidal flow components $U_{s\theta}(\psi_\mathrm{p})$ or using the fact that the current density is also incompressible ($\vec{\nabla}\cdot\vec{J} = 0$) yields
$$K_J(\psi_\mathrm{p}) \equiv \frac{\vec{J}\cdot\vec{\nabla}\theta}{\vec{B}_0\cdot\vec{\nabla}\theta} = \frac{J_\parallel}{B_0} + \frac{I}{B_0^2}\frac{dP_0}{d\psi_\mathrm{p}}.$$

- The constant $K_J$ is determined by multiplying this equation by $B_0^2$ and FSA:
$$K_J = \frac{\langle B_0\, J_\parallel\rangle}{\langle B_0^2\rangle} + \frac{I}{\langle B_0^2\rangle}\frac{dP_0}{d\psi_\mathrm{p}}.$$

- Using this result in the equation for $K_J(\psi_\mathrm{p})$ above yields
$$\boxed{B_0\, J_\parallel = \underbrace{\frac{\langle B_0\, J_\parallel\rangle B_0^2}{\langle B_0^2\rangle}}_{\text{FSA } \parallel \text{ current}} - \underbrace{I\,\frac{dP_0}{d\psi_\mathrm{p}}\left(1 - \frac{B_0^2}{\langle B_0^2\rangle}\right)}_{\text{Pfirsch-Schlüter current}}}\,.$$

- From Ampere's law $[\mu_0\vec{J} = \vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A})]$, the FSA parallel current is
$$\boxed{\mu_0\langle B_0 J_\parallel\rangle = I\langle R^{-2}\rangle\,\Delta^+\psi_\mathrm{p},} \qquad \Delta^+\psi_\mathrm{p} \equiv \frac{I}{\langle R^{-2}\rangle V'}\frac{\partial}{\partial\rho}\left[\left\langle\frac{|\vec{\nabla}\rho|^2}{R^2}\right\rangle\frac{V'}{I}\frac{\partial\psi_\mathrm{p}}{\partial\rho}\right] \simeq \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial\psi_\mathrm{p}}{\partial r}\right).$$

## IIB. Poloidal Flow Is Obtained From Plasma $\|$ Force Balance

- Summing the parallel force balances over species yields (for $\bar{S}_n = 0$)

$$m_i n_{i0} \frac{\partial \langle B_0 V_{i\|} \rangle}{\partial t} \simeq -\langle \vec{B}_0 \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_i \rangle - m_i n_0 \langle \vec{B}_0 \cdot \overline{\tilde{\vec{V}}_i \cdot \vec{\nabla} \tilde{\vec{V}}_i} \rangle + \langle \vec{B}_0 \cdot \overline{\tilde{\vec{J}}_\wedge \times \tilde{\vec{B}}_\perp} \rangle + \langle \vec{B}_0 \cdot \textstyle\sum_s \bar{\vec{S}}_{\rm ps} \rangle.$$

- The poloidal flow is determined mainly by the parallel ion viscous force

$$\langle \vec{B}_0 \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_{i\|} \rangle \simeq m_i n_{i0} \left[ \mu_{i00} U_{i\theta} + \mu_{i01} \frac{-2}{5 n_i T_i} Q_{i\theta} + \cdots \right] \langle B^2 \rangle, \qquad \mu_{i00}, \mu_{i01} \sim \sqrt{\epsilon}\, \nu_i.$$

- For $t > 1/\nu_i \sim 34$ ms, poloidal flow obtained from NCLASS[11] or $\langle \vec{B} \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_{i\|} \rangle \simeq 0$ is

$$U^0_{i\theta}(\psi_{\rm p}) \equiv \frac{\vec{V} \cdot \vec{\nabla} \theta}{\vec{B} \cdot \vec{\nabla} \theta} \simeq -\frac{\mu_{i01}}{\mu_{i00}} \frac{-2}{5 n_i T_i} Q_{i\theta} \simeq \frac{c_{\rm p} I}{q_i \langle B^2 \rangle} \frac{dT_{i0}}{d\psi_{\rm p}} \quad \Longrightarrow \quad \boxed{V_{\rm p} \simeq \frac{1.17}{q_i B} \frac{dT_{i0}}{d\rho} + \mathcal{O}\{\varrho_*^2\}.}$$

- Including all the drives in the parallel plasma force balance above yields[7]

$$\boxed{U_{i\theta}(\psi_{\rm p}) \simeq \underbrace{U^0_{i\theta}(\psi_{\rm p})}_{\text{neoclassical}} - \underbrace{\frac{\langle \vec{B}_0 \cdot (m_i n_{i0} \overline{\tilde{\vec{V}}_i \cdot \vec{\nabla} \tilde{\vec{V}}_i} + \overline{\vec{\nabla} \cdot \overleftrightarrow{\pi}_{i\wedge}}) \rangle}{m_i n_{i0} \mu_{i00} \langle B_0^2 \rangle}}_{\|\text{ Reynolds stress}} + \underbrace{\frac{\langle \vec{B}_0 \cdot \overline{\tilde{\vec{J}}_\wedge \times \tilde{\vec{B}}_\perp} \rangle + \langle \vec{B}_0 \cdot \sum_s \bar{\vec{S}}_{\rm ps} \rangle}{m_i n_{i0} \mu_{i00} \langle B_0^2 \rangle}}_{\|\text{ Maxwell stress + flow sources}}.}$$

- Given the poloidal flow ($\Omega_{*{\rm p}} \equiv I\, U_{i\theta}/R^2$), relation of toroidal flow to $E_\rho$ is:

$$\Omega_{\rm t} \equiv \vec{V} \cdot \vec{\nabla} \zeta = -\left( \frac{d\Phi}{d\psi_{\rm p}} + \frac{1}{n_i q_i} \frac{dp_i}{d\psi_{\rm p}} \right) + \Omega_{*{\rm p}} \quad \Longrightarrow \quad \boxed{V_t \simeq \frac{E_r}{B_{\rm p}} - \frac{1}{n_i q_i B_{\rm p}} \frac{dp_i}{d\rho} + \frac{1.17}{q_i B_{\rm p}} \frac{dT_i}{d\rho}.}$$

[11] W.A. Houlberg, K.C. Shaing, S.P. Hirshman, and M.C. Zarnstorff, "Bootstrap current and neoclassical transport in tokamaks of arbitrary collisionality and aspect ratio," Phys. Plasmas **4**, 3230 (1997).

# IIC. Magnetic Flux Transients Are Important At $\mathcal{O}\{\varrho_*^2\}$

- Poloidal, toroidal magnetic fluxes $\psi_{\rm p}$, $\psi_{\rm t}$ evolve during start-up, addition of current-drives, and approach to steady state on current diffusion times.

- These "slow," $\mathcal{O}\{\varrho_*^2\}$ effects have been negligible in the preceding $\mathcal{O}\{\varrho_*^0, \varrho_*^1\}$ analyses, but need to be included in comprehensive transport equations.

- Using $\vec{B} = \vec{\nabla}\times\vec{A}$ with $\vec{A} = \psi_{\rm t}\vec{\nabla}\theta - \psi_{\rm p}\vec{\nabla}\zeta$ in Faraday's law in the form $\vec{\nabla}\times(\partial\vec{A}/\partial t|_{\vec{x}} - \vec{\nabla}\phi + \vec{E}^A) = \vec{0}$ and FSA of $R^{-2}$ times these equations yields[7]

$$\text{toroidal flux} \qquad \left.\frac{\partial\psi_{\rm t}}{\partial t}\right|_{\vec{x}} = -\,\bar{u}_G\frac{\partial\psi_{\rm t}}{\partial\rho} \equiv \dot{\psi}_{\rm t}, \qquad \bar{u}_G \equiv \langle\vec{u}_G\cdot\vec{\nabla}\rho\rangle = \frac{\langle\vec{B}_{\rm p}\cdot\vec{E}^A\rangle}{\psi_{\rm p}' I\langle R^{-2}\rangle}, \ \text{"grid speed,"}$$

$$\text{poloidal flux} \qquad \left.\frac{\partial\psi_{\rm p}}{\partial t}\right|_{\vec{x}} = \frac{\langle\vec{B}_0\cdot\bar{\vec{E}}^A\rangle}{I\langle R^{-2}\rangle} - \frac{\partial\Psi}{\partial t} - \bar{u}_G\frac{\partial\psi_{\rm p}}{\partial\rho}, \qquad 2\pi\frac{\partial\Psi}{\partial t} \equiv V_{\rm loop}^{\zeta}(t), \ \text{OH solenoid.}$$

- Using $\|$ Ohm's law from p 21 or 32 for $\langle\vec{B}_0\cdot\vec{E}^A\rangle$ and $\mu_0\langle B_0 J_\|\rangle = I\langle R^{-2}\rangle\,\Delta^+\psi_{\rm p}$ yields a diffusion equation for poloidal flux $\psi_{\rm p}$ on a toroidal $\psi_{\rm t}$ flux surface:[7]

$$\boxed{\dot{\psi}_{\rm p} \equiv \left.\frac{\partial\psi_{\rm p}}{\partial t}\right|_{\psi_{\rm t}} = D_\eta\,\Delta^+\psi_{\rm p} - S_\psi,} \qquad D_\eta \equiv \frac{\eta_\|^{\rm nc}}{\mu_0}, \qquad S_\psi = \frac{\partial\Psi}{\partial t} + \frac{1/\sigma_\|}{I\langle R^{-2}\rangle}[\langle\vec{B}_0\cdot(\vec{J}_{\rm bs} + \vec{J}_{\rm CD} + \vec{J}_{\rm dyn})\rangle].$$

## IIC. Plasma Transport Is Relative To Poloidal Flux Surfaces

- **Tokamak plasma properties are determined in terms of poloidal magnetic flux $\psi_{\rm p}$:**

  Grad-Shafranov (ideal MHD equilibrium) equation determines $\psi_{\rm p}(\vec{x})$ given $P(\psi_{\rm p})$ and $I(\psi_{\rm p})$;

  classical and neoclassical transport are determined[12] across poloidal flux surfaces $\psi_{\rm p}$;

  drift-kinetic and gyrokinetic equations use poloidal flux variables and have $f_0 = f_{\rm iM}(\psi_{\rm p})$ — so canonical toroidal angular momentum emerges as a natural constant of motion.

- **Thus, one needs[12] to transform the fluid moment equations from determining density, momentum, energy at a laboratory position $\vec{x}$ to determining them on a poloidal flux surface $\psi_{\rm p}$ — i.e., $\partial n/\partial t|_{\vec{x}} \implies \partial n/\partial t|_{\psi_{\rm p}}$ etc.**

- **However, for low collisionality tokamak plasmas this transformation must first be made in the drift-kinetic (or gyrokinetic) equation, which yields[13] the magnetic-field-diffusion-modified drift-kinetic equation (MDKE → CEKE):**

$$\left.\frac{\partial \bar{f}}{\partial t}\right|_{\psi_{\rm p}} + (\vec{v}_\parallel + \vec{v}_{\rm d})\cdot\vec{\nabla}\bar{f} + \dot{\varepsilon}_{\rm gc}\frac{\partial \bar{f}}{\partial \varepsilon_{\rm gc}} = \bar{\mathcal{C}}\{\bar{f}\} + \mathcal{D}\{\bar{f}\}, \quad \text{in which} \quad \vec{v}_\parallel \equiv v_\parallel \vec{B}/B.$$

- **Here, $\mathcal{D}\{\bar{f}\} \sim D_\eta\,\bar{f}/a^2 \sim \mathcal{O}\{\varrho_*^2\}$ is a second order paleoclassical radial transport operator[12] that results from transformation of DKE equation from $\vec{x}$ to $\psi_{\rm p}$.**

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[12] R.D. Hazeltine, F.L. Hinton and M.N. Rosenbluth, "Plasma transport in a torus of arbitrary aspect ratio," Phys. Fluids **16**, 1645 (1973).
[13] J.D. Callen, Phys. Plasmas **14**, 040701 (2007); **14**, 104702 (2007); **15**, 014702 (2008); **12**, 092512 (2005) — see `www.cae.wisc.edu/∼callen/paleo`.

## IIC. Transform Density Equation With These $\mathcal{O}\{\varrho_*^2\}$ Effects

- FSA paleoclassical transport operator $\mathcal{D} \sim \mathcal{O}\{\varrho_*^2\}$ operating on density is

$$\langle \mathcal{D}\{n_0\}\rangle \equiv -\underbrace{\dot\rho_{\psi_p}\frac{\partial n_0}{\partial \rho}}_{\psi_p\ \text{motion}} + \underbrace{\langle \vec\nabla\cdot n_0 \vec u_G\rangle}_{\psi_t\ \text{motion}} + \underbrace{\frac{1}{V'}\frac{\partial^2}{\partial \rho^2}(V'\bar D_\eta n_0)}_{\text{transport}},$$

$$\dot\rho_{\psi_p} \equiv \frac{\dot\psi_p}{\psi_p'}, \qquad \bar D_\eta \equiv \frac{D_\eta}{\bar a^2}, \qquad \frac{1}{\bar a^2} \equiv \frac{1}{\langle R^{-2}\rangle}\left\langle \frac{|\vec\nabla\rho|^2}{R^2}\right\rangle \gtrsim \frac{1}{a^2}, \qquad \langle \vec\nabla\cdot\vec u_G\rangle = \frac{1}{V'}\frac{\partial V'}{\partial t}\bigg|_\rho .$$

- Including transformation effects, FSA density equation can be written as[7]

$$\frac{1}{V'}\frac{\partial}{\partial t}\bigg|_{\psi_p}(V' n_0) + \underbrace{\dot\rho_{\psi_p}\frac{\partial n_0}{\partial \rho}}_{\psi_p\ \text{motion}} + \underbrace{\frac{1}{V'}\frac{\partial}{\partial\rho}(V'\,\Gamma)}_{\text{transport}} = \underbrace{\langle \bar S_n\rangle}_{\text{sources}},$$

$V' n_0$ is # particles between $\rho$ and $\rho + d\rho$ surfaces, an adiabatic plasma property.

- The total $\mathcal{O}\{\varrho_*^2\}$ particle flux for each species is:

$$\Gamma \equiv \langle \vec\Gamma\cdot\vec\nabla\rho\rangle = \Gamma^a + \Gamma^{na} + \Gamma^a_{pc} = \langle\, [\, \underbrace{n_0(\bar{\vec V}_2 - \vec u_G)}_{\text{collisional}} + \underbrace{\overline{\tilde n_1 \tilde{\vec V}_1}}_{\text{fluctuations}}\, ]\cdot\vec\nabla\rho\,\rangle + \underbrace{\frac{\partial}{\partial\rho}(V'\bar D_\eta n_0)}_{\text{paleoclassical}} .$$

- Note that toroidal flux is basis for radial coordinate $\rho \equiv \sqrt{\psi_t/\pi B_{t0}}$ (units of m) but fluid moments $n, T, \vec V$ are determined on poloidal flux surfaces $\psi_p$.

## IIC. Toroidal Torques From Force Balance Yield Radial Flows

- A key vector identity for determining radial flows is ($\vec{e}_\zeta \equiv R^2 \vec{\nabla}\zeta = R\,\hat{\vec{e}}_\zeta$)

$$\vec{e}_\zeta \cdot \vec{V} \times \vec{B}_0 = -\vec{V} \cdot \vec{e}_\zeta \times \vec{B}_0 = \vec{V} \cdot \vec{\nabla}\psi_{\rm p} \quad \text{— toroidal component of } \vec{V} \times \vec{B}_0 \text{ gives radial flow.}$$

- Thus, the $\vec{e}_\zeta$ component of the force balance shows the particle flux is induced by toroidal torques $T_\zeta \equiv \vec{e}_\zeta \cdot \vec{F}$ on the plasma species by forces $\vec{F}_j$:

$$\vec{e}_\zeta \cdot (nq\vec{V} \times \vec{B}_0 + \textstyle\sum_j \vec{F}_j) = 0 \quad \Longrightarrow \quad \boxed{q\psi_p' \Gamma = -\sum_j \vec{e}_\zeta \cdot \vec{F}_j = -\sum_j T_{\zeta j},} \quad \psi_{\rm p}' \equiv \frac{d\psi_{\rm p}}{d\rho}.$$

- Thus, taking toroidal angular ($\vec{e}_\zeta \,\cdot$) component of the species force balance, averaging over fluctuations and then flux surface averaging yields particle flux:

$$\langle n_0 \bar{\vec{V}}_2 \cdot \vec{\nabla}\psi_{\rm p} \rangle + \langle \overline{\tilde{n}_1 \tilde{\vec{V}}_1} \cdot \vec{\nabla}\psi_{\rm p} \rangle \qquad \text{average plus fluctuation-induced radial particle flux } \Gamma,$$

$$= \frac{1}{q}\left[ -\langle \vec{e}_\zeta \cdot \bar{\vec{R}} \rangle + \langle \vec{e}_\zeta \cdot \vec{\nabla} \cdot \bar{\overleftrightarrow{\pi}} \rangle \right] - n_0 \langle \vec{e}_\zeta \cdot \bar{\vec{E}}^A \rangle \qquad \text{collision-induced particle fluxes,}$$

$$-\langle \vec{e}_\zeta \cdot \overline{\tilde{n}\tilde{\vec{E}}} \rangle - \langle \vec{e}_\zeta \cdot n_0 \overline{\tilde{\vec{V}}_1 \times \tilde{\vec{B}}} \rangle - \frac{1}{q} \langle \vec{e}_\zeta \cdot \bar{\vec{S}}_{\rm p} \rangle + \frac{1}{q} \left( \left. \frac{\partial}{\partial t} \right|_{\vec{x}} [m n_0 \langle \vec{e}_\zeta \cdot \bar{\vec{V}}_1 \rangle] + \langle \vec{\nabla} \cdot \overline{mn\,(\vec{e}_\zeta \cdot \tilde{\vec{V}}_1)\tilde{\vec{V}}_1} \rangle \right), \text{ fluct., inertia.}$$

- After transforming this equation from $\vec{x}$ to $\psi_{\rm p}$ using $\langle \vec{e}_\zeta \cdot \mathcal{D}\{m n_0 \bar{\vec{V}}_1\} \rangle$, it can be solved for the total species particle flux $\Gamma$, which has many contributions.

# Particle Flux Has Many Contributions I: 8 Ambipolar

- The radial particle flux can be written in terms of its various components:[7]

$$\Gamma \equiv \langle \vec{\Gamma}\cdot\vec{\nabla}\rho\rangle \equiv \langle n_{s0}(\bar{\vec{V}}_2 - \vec{u}_G)\cdot\vec{\nabla}\rho\rangle + \langle\overline{\tilde{n}_1 \tilde{\vec{V}}_1\cdot\vec{\nabla}\rho}\rangle - (1/V')(\partial/\partial\rho)\,[\,V'\,\bar{D}_\eta\,n_{s0}] \equiv \Gamma^a + \Gamma^{na} + \Gamma^a_{\rm pc},$$

$$\Gamma = \underbrace{\Gamma_{\rm cl} + \Gamma_{\rm PS} + \Gamma_{\rm bp} + \Gamma_{\rm pc} + \Gamma_{\tilde{E}} + \Gamma_{\rm CD} + \Gamma_{\rm dyn} + \Gamma_{\rm E^A}}_{\Gamma^a + \Gamma^a_{\rm pc},\ 8\ \text{ambipolar (superscript } a)} + \underbrace{\Gamma^{\rm NA}_{\pi\|} + \Gamma_{\pi\perp} + \Gamma_{\rm pol} + \Gamma_{\rm Rey} + \Gamma_{\rm Max} + \Gamma_{\rm JxB} + \Gamma_{\dot{\psi}_{\rm p}} + \Gamma_{\rm S}}_{\Gamma^{na},\ 8\ \text{non-ambipolar (superscript } na)}.$$

- Intrinsically ambipolar fluxes[14] ($\psi'_{\rm p} \equiv d\psi_{\rm p}/d\rho \simeq B_{\rm p} R\, a$):

$$\Gamma_{\rm cl} = \left\langle \frac{\vec{B}_0\times\vec{\nabla}\rho}{B_0^2}\cdot\frac{\bar{\vec{R}}_{s\perp}}{q_s}\right\rangle = -\frac{n_{e0}}{\sigma_\perp}\left\langle\frac{|\vec{\nabla}\rho|^2}{B_0^2}\right\rangle\frac{dP_0}{d\rho}, \qquad D_{\rm cl} \simeq \frac{n_{e0}(T_e+T_i)}{\sigma_\perp\langle B_0^2\rangle} \simeq \nu_e \varrho_e^2, \qquad \text{classical,}$$

$$\Gamma_{\rm PS} = -\frac{n_{e0} I^2}{\sigma_\|\,\psi_{\rm p}'^2}\left\langle\frac{1}{B_0^2}\left(1-\frac{B_0^2}{\langle B_0^2\rangle}\right)^2\right\rangle\frac{dP_0}{d\rho}, \qquad D_{\rm PS} \simeq \frac{2\sigma_\perp}{\sigma_\|} q^2 D_{\rm cl} \sim q^2 D_{\rm cl}, \qquad \text{Pfirsch-Schlüter,}$$

$$\Gamma_{\rm bp} = \frac{I}{e\psi'_{\rm p}\langle B_0^2\rangle}\,\langle\vec{B}_0\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_{e\|}\rangle, \qquad D_{\rm bp} \simeq \mu_e \varrho_{ep}^2 \sim \frac{q^2}{\epsilon^{3/2}}\,D_{\rm cl}, \qquad \text{banana-plateau,}$$

$$\Gamma_{\rm pc} = -\left(\bar{D}_\eta\frac{dn_{e0}}{d\rho} + n_{e0}V_{\rm pc}\right), \quad V_{\rm pc} \equiv \frac{1}{V'}\frac{\partial}{\partial\rho}(V'\bar{D}_\eta), \quad D_\eta \equiv \frac{\eta_\|^{\rm nc}}{\mu_0} \sim \frac{D_{\rm cl}}{\beta_e}, \qquad \text{paleoclassical,}$$

$$\Gamma_{\tilde{E}} = \langle\overline{\tilde{n}\tilde{\vec{V}}_E}\cdot\vec{\nabla}\rho\rangle - (I/\psi'_{\rm p})\,\langle\vec{B}_0\cdot\overline{\tilde{n}\tilde{\vec{E}}}\,/B_0^2\rangle, \qquad \text{fluctuation-induced density flux,}$$

$$\Gamma_{\rm CD} + \Gamma_{\rm dyn} = [(n_{e0}I)/(\sigma_\|\,\psi'_{\rm p}\langle B_0^2\rangle)]\langle\vec{B}_0\cdot(\vec{J}_{\rm CD} + \vec{J}_{\rm dyn})\rangle, \qquad \text{current drive, dynamo effects,}$$

$$\Gamma_{\rm E^A} = -\,n_{e0}\left[\langle\vec{e}_\zeta\cdot\bar{\vec{E}}^A\rangle(1 - I^2\langle 1/R^2\rangle/\langle B_0^2\rangle) - I\langle\vec{B}_p\cdot\bar{\vec{E}}^A\rangle/\langle B_0^2\rangle\right]/\psi'_{\rm p}, \qquad \bar{\vec{E}}^A\times\vec{B}_{\rm p}/B_0^2\ \text{pinch.}$$

[14]K.C. Shaing, S.P. Hirshman, and J.D. Callen, Phys. Fluids **29**, 521 (1986); K.C. Shaing, Phys. Fluids **29**, 2231 (1986).

# Particle Flux Has Many Contributions II: 8 Non-ambipolar[7]

- Non-ambipolar fluxes (Γ's here are multiplied by $\psi'_{\mathrm{p}} \equiv d\psi_{\mathrm{p}}/d\rho \simeq B_{\mathrm{p}} R\, a$):

$$\Gamma_{\pi\|}^{\mathrm{NA}} = \frac{1}{q_s}\langle \vec{e}_\zeta \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_{s\|}^{\mathrm{NA}} \rangle \simeq \frac{m_i n_{i0} \langle R^2 \rangle\, \mu_{i\,\mathrm{t}}}{q_i} \left( \frac{\delta \tilde{B}_{\mathrm{eff}}}{B_0} \right)^2 (\Omega_{\mathrm{t}} - \Omega_*), \quad \Omega_* \simeq \frac{c_{\mathrm{p}} + c_{\mathrm{t}}}{q_i} \frac{dT_i}{d\psi_{\mathrm{p}}}, \qquad \text{NTV},$$

$$\Gamma_{\pi\perp} = \frac{1}{q_s}\langle \vec{e}_\zeta \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_{s\perp} \rangle \simeq \frac{1}{q_i} \left\langle \vec{e}_\zeta \cdot \vec{\nabla} \cdot (\overleftrightarrow{\pi}_{i\perp}^{\mathrm{cl}} + \overleftrightarrow{\pi}_{i\perp}^{\mathrm{nc}} + \overleftrightarrow{\pi}_{i\perp}^{\mathrm{pc}}) \right\rangle \sim -\chi_{\mathrm{t}} \nabla^2 \Omega_{\mathrm{t}}, \;\; \chi_{\mathrm{t}\,i} \sim (1 + 0.1 q^2) \nu_i \varrho_i^2 + D_\eta,$$

$$\Gamma_{\mathrm{pol}} = \frac{1}{q_s V'} \left. \frac{\partial}{\partial t} \right|_{\psi_{\mathrm{p}}} \Big( V' m_s n_{s0} \langle \vec{e}_\zeta \cdot \bar{\vec{V}}_s \rangle \Big), \qquad \text{ion polarization flow when } (\partial \Omega_{\mathrm{t}}/\partial t) \neq 0,$$

$$\Gamma_{\mathrm{Rey}} = \frac{1}{q_s V'} \frac{\partial}{\partial \rho} (V' \Pi_{s\rho\zeta}), \quad \boxed{\Pi_{s\rho\zeta} \equiv m_s n_{s0} \langle \overline{(\vec{\nabla}\rho \cdot \tilde{\vec{V}}_s)(\tilde{\vec{V}}_s \cdot \vec{e}_\zeta)} \rangle + \langle \vec{\nabla}\rho \cdot \overline{\overleftrightarrow{\pi}_{s\wedge}} \cdot \vec{e}_\zeta \rangle}, \quad \text{Reynolds stress},$$

$$\Gamma_{\mathrm{Max}} = -\langle \vec{e}_\zeta \cdot \overline{\widetilde{n_1 \vec{V}_1} \times \tilde{\vec{B}}} \rangle \simeq \frac{1}{e} \langle \vec{e}_\zeta \cdot \overline{\tilde{\vec{J}} \times \tilde{\vec{B}}} \rangle = \frac{1}{e\mu_0} \langle \vec{e}_\zeta \cdot \overline{\tilde{\vec{B}} \cdot \vec{\nabla} \tilde{\vec{B}}} \rangle, \qquad \text{Maxwell stress},$$

$$\Gamma_{\mathrm{JxB}} \simeq \frac{1}{e} \langle \vec{e}_\zeta \cdot \overline{\delta \vec{J}_{\|m/n} \times \delta \vec{B}_{\perp m/n}} \rangle \sim \delta[\rho - \rho_{m/n}] \, \frac{c_{\mathrm{A}\theta}}{e} \, \frac{\omega\, m_i n_{i0}\, R}{\Delta'^2 + (\omega \tau_\delta)^2} \, \frac{\delta B^2_{\rho\, m/n}}{B_0^2}, \quad \text{FE-induced res. layer},$$

$$\Gamma_{\dot{\psi}_{\mathrm{p}}} = \frac{\dot{\rho}_{\psi_{\mathrm{p}}}}{q_s} \frac{\partial}{\partial \rho} (m_s n_{s0} \langle \vec{e}_\zeta \cdot \bar{\vec{V}}_s \rangle), \qquad \psi_{\mathrm{p}} \text{ transients},$$

$$\Gamma_{s\mathrm{S}} = -\frac{1}{q_s} \langle \vec{e}_\zeta \cdot \vec{S}_{\mathrm{p}s} \rangle, \qquad \text{momentum sources (e.g., NBI, CD).}$$

## IIC. Setting To Zero Radial Current Obtained By Summing Particle Fluxes Over Species Yields Toroidal Torque Balance

- Sum radial species currents to obtain net radial plasma current:

$$\langle \vec{J}\cdot\vec{\nabla}\rho\rangle \equiv \sum_s q_s \left(\Gamma_s^{a} + \Gamma_{\text{spc}}^{a} + \Gamma_s^{an}\right) = \sum_s q_s\,\Gamma_s^{an} \quad \text{— sum of non-ambipolar currents.}$$

- Charge continuity equation on a $\psi_{\rm p}$ surface is obtained by summing $q_s$ times density equations over species is ($\dot{\rho}_{\psi_{\rm p}} = 0$ and $\sum_s q_s\langle \bar{S}_{ns}\rangle = 0$ for simplicity)

$$\frac{1}{V'}\frac{\partial}{\partial t}\bigg|_{\psi_{\rm p}}(V'\langle\rho_q\rangle) + \frac{1}{V'}\frac{\partial}{\partial\rho}(V'\langle\vec{J}\cdot\vec{\nabla}\rho\rangle) = 0.$$

- For quasineutrality at all $t$ this charge continuity equation requires $\langle\vec{J}\cdot\vec{\nabla}\rho\rangle = 0$.

- Setting $\langle\vec{J}\cdot\vec{\nabla}\rho\rangle$ to zero yields comprehensive toroidal torque balance equation[7] for the total toroidal plasma angular momentum density $L_{\rm t} \equiv m_i n_{i0}\,\langle R^2\rangle\,\Omega_{\rm t}$:

$$\underbrace{\frac{1}{V'}\frac{\partial}{\partial t}\bigg|_{\psi_{\rm p}}(V'L_{\rm t})}_{\text{inertia}} \simeq -\underbrace{\langle\vec{e}_\zeta\cdot\vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}{}_{i\|}^{\rm NA}\rangle}_{\text{NTV from } \tilde{B}_\|} - \underbrace{\langle\vec{e}_\zeta\cdot\vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}_{i\perp}\rangle}_{\text{cl, neo, paleo}} - \underbrace{\frac{1}{V'}\frac{\partial}{\partial\rho}(V'\Pi_{i\rho\zeta})}_{\text{Reynolds stress}} + \underbrace{\langle\vec{e}_\zeta\cdot\overline{\tilde{\vec{J}}\times\tilde{\vec{B}}}\rangle}_{\text{res. FE, Max}} - \underbrace{\dot{\rho}_{\psi_{\rm p}}\frac{\partial L_{\rm t}}{\partial\rho}}_{\psi_{\rm p}\text{ motion}} + \underbrace{\langle\vec{e}_\zeta\cdot\textstyle\sum_s\bar{\vec{S}}_{{\rm p}s}\rangle}_{\text{sources}}.$$

## IIC. Toroidal Rotation Equation Includes Many Different Effects

- Equation for the toroidal angular momentum density $L_\mathrm{t} \equiv m_i n_{i0} \langle R^2 \rangle \Omega_\mathrm{t}$ is:[7]

$$\underbrace{\frac{1}{V'}\frac{\partial}{\partial t}\bigg|_{\psi_\mathrm{p}}(V' L_\mathrm{t})}_{\text{inertia}} \simeq -\underbrace{\langle \vec{e}_\zeta \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_{i\parallel}^{\mathrm{NA}} \rangle}_{\text{NTV from } \tilde{B}_\parallel} - \underbrace{\langle \vec{e}_\zeta \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_{i\perp} \rangle}_{\text{cl, neo, paleo}} - \underbrace{\frac{1}{V'}\frac{\partial}{\partial \rho}(V' \Pi_{i\rho\zeta})}_{\text{Reynolds stress}} + \underbrace{\langle \vec{e}_\zeta \cdot \overline{\tilde{\vec{J}} \times \tilde{\vec{B}}} \rangle}_{\text{res. FE, Max}} - \underbrace{\dot{\rho}_{\psi_\mathrm{p}} \frac{\partial L_\mathrm{t}}{\partial \rho}}_{\psi_\mathrm{p} \text{ motion}} + \underbrace{\langle \vec{e}_\zeta \cdot \textstyle\sum_s \vec{S}_{\mathrm{p}s} \rangle}_{\text{sources}}.$$

- Neoclassical toroidal viscous (NTV) damping (to be discussed in next lecture) by 3-D non-axisymmetric ($^{\mathrm{NA}}$) $\delta \vec{B}$ fields drives $\Omega_\mathrm{t} \to \Omega_*$ via

$$-\langle \vec{e}_\zeta \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_{i\parallel}^{\mathrm{NA}} \rangle \simeq -m_i n_{i0} \langle R^2 \rangle \mu_{i\,\mathrm{t}} \left( \frac{\delta B_{\parallel \mathrm{eff}}}{B_0} \right)^2 (\Omega_\mathrm{t} - \Omega_*), \quad \Omega_* \simeq \frac{c_\mathrm{p} + c_\mathrm{t}}{q_i} \frac{dT_i}{d\psi_\mathrm{p}}, \text{ offset velocity.}$$

Damping frequency $\mu_{i\,\mathrm{t}} \sim 1/\omega_E^2$ in low $\nu$ regime yields max NTV torque where $|\vec{E} \times \vec{B}_0| \to 0$.

- Collisional $\perp$ viscous stresses are dominated by paleoclassical processes:

$$-\langle \vec{e}_\zeta \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_{i\perp} \rangle \simeq -\frac{1}{V'}\frac{\partial}{\partial \rho}\left[ V' \left( \bar{D}_\eta \frac{\partial L_\mathrm{t}}{\partial \rho} + L_\mathrm{t} V_\mathrm{pc} \right) \right], \quad \text{only significant for } T_e \lesssim B_0^{2/3} \bar{a}^{1/2} \lesssim 5 \text{ keV}.$$

- Microtubulence-induced ion Reynolds stresses cause radial transport of $L_\mathrm{t}$:

$$\Pi_{i\rho\zeta} \equiv m_i n_{i0} \langle \overline{(\vec{\nabla}\rho \cdot \tilde{\vec{V}}_i)(\tilde{\vec{V}}_i \cdot \vec{e}_\zeta)} \rangle + \langle \vec{\nabla}\rho \cdot \overline{\overleftrightarrow{\pi}_{i\wedge}} \cdot \vec{e}_\zeta \rangle \;\sim\; \underbrace{-\chi_\mathrm{t} \frac{\partial L_\mathrm{t}}{\partial \rho}}_{\text{diffusion}} + \underbrace{L_\mathrm{t} V_\mathrm{pinch}}_{\text{pinch}} + \underbrace{\Pi_{i\rho\zeta}^{\mathrm{RS}}}_{\text{residual stress}},$$

which in the core of a tokamak usually balances momentum source $\langle \vec{e}_\zeta \cdot \sum_s \vec{S}_{\mathrm{p}s} \rangle$ from NBI.

## IIC. Toroidal Rotation Determines Radial Electric Field Required For Net Ambipolar Radial Particle Flux

- The radial electric field determined from toroidal rotation $\Omega_{\rm t} \equiv L_{\rm t}/(m_i n_{i0}\langle R^2\rangle)$ is:

$$E_\rho \equiv -|\vec{\nabla}\rho|\,\frac{d\Phi_0}{d\rho} \simeq |\vec{\nabla}\rho|\left(\Omega_{\rm t}\,\psi_{\rm p}' + \frac{1}{n_{i0}q_i}\frac{dp_{i0}}{d\rho} - \frac{c_{\rm p}}{q_i}\frac{dT_{i0}}{d\rho}\right), \qquad |\vec{\nabla}\rho| \text{ varies with } \theta.$$

- The resultant $E_\rho$ (or $\Omega_{\rm t}$) causes the electron and ion non-ambipolar radial particle fluxes to become equal (i.e., ambipolar):

$$\Gamma_e^{na}(E_\rho) = Z_i\,\Gamma_i^{na}(E_\rho) \implies \langle \vec{J}\cdot\vec{\nabla}\rho\rangle = 0 \implies L_{\rm t} \text{ (i.e., } \Omega_{\rm t}, \text{ or } E_\rho) \text{ equation.}$$

- Thus, the net ambipolar particle flux is sum of $\Gamma^a + \Gamma^a_{\rm pc}$ and $\Gamma^{na}(E_\rho)$, which is usually easiest to evaluate for electrons since $\langle \vec{J}\cdot\vec{\nabla}\rho\rangle \simeq \Gamma_i^{na}(E_\rho) \simeq 0$, which is usually called the "ion root:"

$$\Gamma \equiv \Gamma_e^{\rm net} \equiv \underbrace{\Gamma_e^a + \Gamma_{e{\rm pc}}^a}_{\substack{\text{intrinsically}\\ \text{ambipolar}}} + \underbrace{\Gamma_e^{na}(E_\rho)}_{\substack{\text{non-ambipolar}\\ \overset{E_\rho}{\Longrightarrow}\ \text{ambipolar}}} = \Gamma_i^{\rm net}.$$

# III. Resultant Transport Equations Can Now Be Specified

- Density (assuming for simplicity the particle source $\langle \bar{S}_n \rangle$ is ambipolar):

$$\boxed{\frac{1}{V'}\frac{\partial}{\partial t}\bigg|_{\psi_\mathrm{p}}(V' n_e) + \dot{\rho}_{\psi_\mathrm{p}}\frac{\partial n_e}{\partial \rho} + \frac{1}{V'}\frac{\partial}{\partial \rho}\left[V'\,\Gamma_e^\mathrm{net}(E_\rho)\right] = \langle \bar{S}_n \rangle,} \quad \text{here } n_e \equiv n_{e0}, \quad \dot{\rho}_{\psi_\mathrm{p}} \equiv \frac{\dot{\psi}_\mathrm{p}}{\psi'_\mathrm{p}},$$

$$\Gamma \equiv \Gamma_e^\mathrm{net}(E_\rho) \equiv \Gamma_e^a + \Gamma_\mathrm{epc}^a + \Gamma_e^{na}(E_\rho) \simeq \underbrace{\Gamma_\mathrm{bp} + \Gamma_\mathrm{pc}}_{\text{collision-induced}} + \underbrace{\Gamma_{e\,\tilde{E}} + \Gamma_{e\,\mathrm{Rey}}(E_\rho) + \Gamma_{e\,\mathrm{Max}}(E_\rho)}_{\text{fluctuations}}$$

$$\simeq \underbrace{\overline{\langle \tilde{n}_e \vec{\tilde{V}}_{\tilde{E}} \cdot \vec{\nabla}\rho \rangle}}_{\text{micro-turbulence}} + \Gamma_\mathrm{bp} \underbrace{- \bar{D}_\eta \frac{\partial n_e}{\partial \rho} - n_e V_\mathrm{pc}}_{\text{paleo diffusion \& pinch}} \underbrace{- \frac{1}{\psi'_p}\langle \vec{e}_\zeta \cdot n_e \overline{\vec{\tilde{V}}_e \times \vec{\tilde{B}}} \rangle}_{e\text{ Maxwell stress}}.$$

- For toroidal rotation $\Omega_\mathrm{t} \equiv L_\mathrm{t}/(m_i n_{i0}\langle R^2 \rangle)$

  see p 41, 42 for $L_\mathrm{t} \equiv m_i n_{i0} \langle R^2 \rangle \Omega_t$ equation (and preceding viewgraph for $E_\rho$).

- Collisional entropy (s) evolution equations for electrons and ions have forms of heat fluxes similar to particle fluxes, but without the ambipolar constraint — see reference 8 cited on p 27.

# Tokamak Transport Equations Include Many Effects

- **With sources of $n$, $L_{\rm t} \equiv \rho_m \langle R^2 \rangle \Omega_{\rm t}$ and $p_s$, transport equations are[9]**

$$\text{density} \qquad \frac{1}{V'}\frac{\partial}{\partial t}\bigg|_{\psi_{\rm p}} n_e V' + \dot{\rho}_{\psi_{\rm p}} \frac{\partial n_e}{\partial \rho} + \frac{1}{V'}\frac{\partial}{\partial \rho}(V' \Gamma) \;=\; \langle \overline{S}_n \rangle,$$

$$\text{tor. mom.} \qquad \frac{1}{V'}\frac{\partial}{\partial t}\bigg|_{\psi_{\rm p}} L_{\rm t} V' + \dot{\rho}_{\psi_{\rm p}} \frac{\partial L_{\rm t}}{\partial \rho} + \frac{1}{V'}\frac{\partial}{\partial \rho}(V' \overline{\Pi}_{\rho\zeta}) \;=\; \langle \vec{e}_\zeta \cdot \Big( \overline{\vec{J}\times\vec{B}} - \vec{\nabla}\cdot \overline{\overleftrightarrow{\Pi}} + \textstyle\sum_s \overline{\vec{S}}_{{\rm p}s} \Big) \rangle,$$

$$\text{energy} \qquad \frac{3}{2} p_s \frac{\partial}{\partial t}\bigg|_{\psi_{\rm p}} \ln p_s V'^{5/3} + \frac{3}{2}\, \dot{\rho}_{\psi_{\rm p}} \frac{\partial p_s}{\partial \rho} + \frac{1}{V'}\frac{\partial}{\partial \rho}(V' \Upsilon_s) + \langle \vec{\nabla}\cdot \vec{q}^{\,\rm pc}_{s*} \rangle \;=\; \overline{Q}_{s\,\rm net}.$$

- **In these comprehensive tokamak plasma transport equations:**

$V' \equiv dV/d\rho$ (m$^2$) is the radial derivative of the volume $V(\rho)$ (m$^3$) of the $\rho$ (m) surface and

$V' n_e \equiv dN/d\rho$ and $V' L_{\rm t}$ are # of particles $N$ and plasma toroidal angular momentum between $\rho$ and $\rho + d\rho$ flux surfaces, which are both adiabatic (isentropic) properties;

similarly, $\ln p_s V'^{5/3}$ is collisional entropy density between the $\rho$ and $\rho + d\rho$ flux surfaces;

further, $\dot{\rho}_{\psi_{\rm p}} \equiv -\dot{\psi}_{\rm p}/\psi'_{\rm p}$ takes account of $\psi_{\rm p}$ surface motion relative to the $\psi_{\rm t}$-based $\rho$;

$\overline{\overleftrightarrow{\Pi}}_{\rho\zeta} \equiv \sum_s \overline{\overleftrightarrow{\pi}}_{s\rho\zeta}$, $\quad \overline{\overleftrightarrow{\pi}}_{s\rho\zeta} = m_s n_s \langle \vec{\nabla}\rho \cdot \overline{\tilde{\vec{V}}_s \tilde{\vec{V}}_s} \cdot \vec{e}_\zeta \rangle + \langle \vec{\nabla}\rho \cdot \overline{\overleftrightarrow{\pi}}_{\wedge s} \cdot \vec{e}_\zeta \rangle$ is $\mu$turbulence-induced Reynolds stress;

and $\langle \vec{\nabla}\cdot \vec{q}^{\,\rm pc}_{s*} \rangle = -\dfrac{M_s}{V'}\dfrac{\partial^2}{\partial \rho^2}\Big( V' \overline{D}_\eta \dfrac{3}{2} p_s \Big) + \dfrac{3}{2}\, \dot{\rho}_{\psi_*} \dfrac{\partial p_s}{\partial \rho}$ is due to[13] paleoclassical helical electron heat transport.

Some $\langle \vec{e}_\zeta \cdot \overline{\vec{J}\times\vec{B}} \rangle$ and $\langle \vec{e}_\zeta \cdot \vec{\nabla}\cdot \overline{\overleftrightarrow{\Pi}} \rangle$ closures for small 3-D fields have been obtained and validated.[15]

[15] J.D. Callen, "Effects of 3D magnetic perturbations on toroidal plasmas," Nucl. Fusion **51**, 094026 (2011).

# This Approach Is New And Has Some Consequences

- **Key differences from usual approaches for plasma transport equations are:**

  **first solve for electrons & ion flows within flux surfaces $\to \parallel$ Ohm's law & poloidal ion flow;**

  **derivation of non-ambipolar density fluxes and toroidal rotation ($\to \boldsymbol{E}_\rho$) are naturally joined;**

  **comprehensive transport equations are obtained for $\Omega_{\rm t}$ ($\to \boldsymbol{E}_\rho$) and $\psi_{\rm p}$, as well as usual $\boldsymbol{n}_e$, $\boldsymbol{p}_s$;**

  **effects of micro-turbulence on $\parallel$ Ohm's law (p 32), poloidal ion flow (p 34), particle fluxes (p 39, 40), momentum transport (p 41, 42) and $\boldsymbol{E}_\rho$ (p 43, 44) are all included self-consistently;**

  **fluctuation-induced density flux is obtained from electron $\overline{\langle \tilde{n}_e \vec{\tilde{V}}_{\tilde{E}} \cdot \vec{\nabla} \rho \rangle}$ plus Rey., Max. stresses;**

  **source effects (e.g., NBI momentum input and $\vec{J}_{\rm CD}$) are included self-consistently;**

  **poloidal field transients ($\dot{\psi}_p \neq 0$) and current diffusion time scale effects are included; and**

  **net transport equations follow naturally from extended two-fluid moment equations and hence are consistent with M3D, NIMROD, JOREK etc. extended MHD code frameworks.**

- **Some new attributes and elements of this approach are:**

  **radial electric field is determined self-consistently and enforces ambipolar density transport;**

  **micro-turbulence should be determined from Chapman-Enskog kinetic equation (CEKE) — so closures and transport they induce are consistent with these FSA transport equations;**

  **paleoclassical $\boldsymbol{n}$, $\Omega_{\rm t}$ ($\to \boldsymbol{E}_\rho$), $\boldsymbol{p}_s$ diffusion and pinch effects are included naturally; and**

  **poloidal flux transients ($\dot{\psi}_{\rm p} \neq 0$) induce radial motion of $\boldsymbol{n}$, $\Omega_{\rm t}$ ($\to \boldsymbol{E}_\rho$), $\boldsymbol{p}_s$.**

# SUMMARY: Status, Issues And Research Topics

- ***Extended MHD***: Collisional and closure moments needed to close magnetized fluid equations were identified, and model discussed.

- ***Tokamak Extended MHD:*** The tokamak axisymmetric ideal MHD equilibrium was discussed. Also, viscous forces and their effects on the parallel Ohm's law and poloidal flows in a tokamak were discussed. Finally, the fluid moment equations were transformed to magnetic flux coordinates, flux surface averaged and used to obtain the tokamak plasma transport equations for $n_e$, $\Omega_{\rm t}$ and $p_s$.

- ***Status***: Extended MHD and tokamak transport equations are still being developed for applications where new closures are required.

- Possible ***research topics*** in these areas are development of

  procedures and algorithms for obtaining "local" collisional and closure moments for extended MHD when only flux surface averages are available,

  procedures and algorithms for solving the CEKE for $F_s$ in the presence of microturbulence that yield needed closures for new transport equations,

  and useful procedures, closures and algorithms in the vicinity of X points near and outside a divertor magnetic separatrix.

# Subjects To Be Covered In Final Lecture 4

- **Tokamak plasma transport modeling (Chapter 5):**

  **there are many effects in tokamak plasma transport equations — $\psi_{\rm p}$ transients, collision- and microturbulence-induced transport, sources and sinks,**

  **plus toroidal $\vec{J}\times\vec{B}$ and viscous forces $\vec{\nabla}\cdot\overleftrightarrow{\Pi}$ caused by small 3-D fields, many recently developed examples of which will be discussed.**

- **New strategy for achieving comprehensive "grand unified tokamak simulations" (GUTS) that can provide the "predictive capability" needed for behavior of plasmas in ITER (Chapter 6):**

  **1) use extended MHD to check macrostability and determine $\vec{B}$ field with plasma responses on collisionless through (via closures) collisional time scales,**

  **2) solve relevant drift-kinetic/gyrokinetic Chapman-Enskog kinetic equation for $F_s$ in this $\vec{B}$ field, for both collisional and microinstability processes,**

  **3) obtain collisional and closure moments needed for extended MHD and the resultant comprehensive tokamak plasma transport equations,**

  **4) solve tokamak plasma transport equations simultaneously for $n_e$, $\Omega_{\rm t}$ ($E_\rho$), $p_s$,**

  **5) and then iterate back through steps 1) to 4).**