Response to "Comment on 'Derivation of paleoclassical key hypothesis' " [Phys. Plasmas 15, 014701 (2008)]

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The comment[1] mainly challenges whether the derivation of the key hypothesis of the paleoclassical model in Ref. 2 is a systematic, "genuine" derivation or just a mathematical restatement of the key hypothesis. The *key hypothesis of the paleoclassical model* of radial electron heat transport is that[3] electron guiding centers diffuse radially along with thin annuli of poloidal magnetic flux in resistive, current-carrying axisymmetric toroidal plasmas, e.g., tokamak plasmas.

While the key hypothesis of the paleoclassical model was only motivated phenomenologically in the original toroidal paper,[3] Ref. 2 presents a derivation of it. The derivation is based on transforming (mapping) the drift-kinetic equation from laboratory to poloidal magnetic flux coordinates in situations where the poloidal flux obeys a slowly evolving diffusion equation, determining the mathematical characteristic curves (effective particle trajectories) including the slow poloidal flux diffusion effects, and taking account of the resultant slowly radially diffusing particle trajectories via a Fokker–Planck spatial diffusion operator. A simpler model that illustrates the key points involved in the derivation was presented in Ref. 4. The salient points are:[4] "(1) particles oscillate rapidly about a ψ surface; (2) ψ obeys a slowly evolving diffusion equation; and (3) further analysis of the kinetic equation is to be carried out on and relative to ψ surfaces"—all of which are satisfied by the derivation in Ref. 2.

The starting point for the derivation of both neoclassical[5,6] and paleoclassical[3] transport is the drift-kinetic equation in poloidal flux coordinates,[1,2,5]

$$\left.\frac{\partial f}{\partial t}\right|_{\psi} + \left.\frac{\partial \psi}{\partial t}\right|_{\mathbf{x}}\frac{\partial f}{\partial \psi} + (\mathbf{v}_{\parallel} + \mathbf{v}_D)\cdot\nabla f + \dot{\varepsilon}_g\frac{\partial f}{\partial \varepsilon_g} = \mathcal{C}\{f\}, \quad (1)$$

in which $f(\psi,\theta,\mu,\varepsilon_g,t)$ is the gyroaveraged distribution function. The poloidal magnetic flux function $\psi(\rho,t)$ is governed by[2,3]

$$\left.\frac{\partial \psi}{\partial t}\right|_{\mathbf{x}} + \bar{u}_G\frac{\partial \psi}{\partial \rho} = D_\eta \Delta^{+}\psi - S_\psi. \quad (2)$$

Here, ρ is a dimensionless radial coordinate based on the toroidal magnetic flux ψ_t which is stationary with respect to the laboratory ($\mathbf{x}$) coordinate system except for its "grid speed" $\bar{u}_G$ (i.e., $d\psi_t/dt \equiv \partial\psi_t/\partial t|_{\mathbf{x}} + \bar{u}_G\partial\psi_t/\partial\rho = 0$), $D_\eta \equiv \eta_{\parallel}^{\rm nc}/\mu_0 \sim \nu_e(c/\omega_p)^2$ is the magnetic field diffusivity induced by the parallel neoclassical resistivity $\eta_{\parallel}^{\rm nc}$ in the plasma, Δ^{+} is a second order cylindrical-type differential operator in ρ and S_ψ represents the sources of poloidal flux (due to the Ohmic heating transformer, bootstrap current, and noninductive sources).

The comment[1] states that "The correct procedure for solving Eq. (1), including the $\partial\psi/\partial t$ term, by systematic expansion in the small Larmor radius parameter (labeled by m/e) can be found in Ref. 5, and the ordering employed in Ref. 2 is identical to that in Ref. 5." The reasons why the previous solution procedure is incomplete when ψ satisfies a slowly evolving diffusion equation, which is second order in the gyroradius (i.e., $\mathcal{O}\{\delta^2\}$ in which $\delta \sim \varrho/L_\perp \sim m/e$) since the electromagnetic skin depth $\delta_e \equiv c/\omega_p$ is of order the gyroradius $\varrho \equiv v_\perp/\omega_c = mv_\perp/eB$, are discussed in the paragraphs after Eqs. (5) and (19) in Ref. 2. The basic point is that the diffusion term $D_\eta\Delta^{+}\psi$ with $D_\eta \equiv \eta/\mu_0 \sim \nu_e\delta_e^2$ in Eq. (2) represents a second order term (in the small gyroradius expansion) whose effect is to cause slow $\mathcal{O}\{\delta^2\}$, secular radial diffusion of the particle guiding centers—as determined in Eqs. (6)–(13) of Ref. 2, Eqs. (7)–(12) of Ref. 4, and Eqs. (6)–(15) below.

Reference 5 specifically states [after its Eq. (10)]: "Here, the second-order terms are only required to derive neoclassical moment equations." That is, only the first order terms and the zeroth and first order distribution functions were used and determined in obtaining the second order neoclassical transport fluxes in Ref. 5; a complete analysis including all the effects of the $\mathcal{O}\{\delta^2\}$ terms from $\partial\psi/\partial t|_{\mathbf{x}}$ and their direct contribution to the second order distribution function and transport were not determined. As discussed in the paragraph after Eq. (19) in Ref. 2, all the $\mathcal{O}\{\delta^2\}$ effects (temporal variation, grid speed $\bar{u}_G$, as well as diffusion due to D_η) from the equation for $\partial\psi/\partial t|_{\mathbf{x}}$ are needed to produce a complete magnetic-field-diffusion-modified drift-kinetic equation (MDKE) on ψ surfaces whose velocity-space moments yield the proper fluid moment equations[6,7] for an evolving axisymmetric toroidal plasma. In particular, the fluid moments in Eqs. (13)–(15) in Ref. 5 do not include the "grid speed" effects that were later included in neoclassical analyses by Hinton and Hazeltine[6] and Hirshman and Jardin.[7] The grid speed effects are needed in the fluid moment equations in order to be determining the neoclassical radial transport fluxes on poloidal magnetic flux surfaces, but relative to the toroidal magnetic flux surfaces that move with the grid speed $\bar{u}_G$ relative to laboratory coordinates. The $\bar{u}_G$ grid speed effects were included in Refs. 6 and 7 within a fluid moment context by taking flux surface averages of the fluid moment

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equations assuming $\partial\psi/\partial t|_{\mathbf{x}}+\bar{u}_G\partial\psi/\partial\rho=0$, i.e., assuming advective motion of the ψ surfaces and neglecting the effects of the $\mathcal{O}\{\delta^2\}$ diffusion and source terms on the right-hand side of Eq. (2). The key new element[3] of the paleoclassical model is to include the effects of all the $\mathcal{O}\{\delta^2\}$ terms in Eq. (2) and to do so within a drift-kinetic rather than fluid moment plasma description.

Perhaps the most fundamental question about the paleoclassical model is: How can the diffusion of particle trajectories be derived from the fundamental equations of motion? Thus, as an alternative procedure to integration of the drift-kinetic equation in Eq. (1) along its mathematical characteristic curves,[2] consider deriving particle trajectories directly from Newton's second law with a Lorentz force, $md\mathbf{v}/dt|_{\mathbf{x}}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$. Expanding the various terms in this equation in the smallness of the gyroradius relative to the magnetic field inhomogeneity scale lengths and averaging over the fast gyromotion time scale, one obtains the usual[6] guiding center motion equation

$$\partial\mathbf{x}_g/\partial t|_{\mathbf{x}}=\mathbf{v}_\parallel+\mathbf{v}_D. \tag{3}$$

Here, the time derivative of the guiding center position $\mathbf{x}_g$ is taken at a constant laboratory coordinate position $\mathbf{x}$. Also, $\mathbf{v}_\parallel$ and $\mathbf{v}_D$ are the parallel and drift velocities of the guiding center, which are of zeroth and first order in the usual small gyroradius expansion, respectively. (Second and higher order effects on these velocities can be included as well,[8] but they are not important here.)

Next, transform (map) the guiding center motion equation in Eq. (3) from laboratory ($\mathbf{x}$) to poloidal magnetic flux coordinates (ψ,θ,ζ), which, since the toroidal angle ζ is invariant and the poloidal angle θ is nearly so,[2] yields $\partial\mathbf{x}_g/\partial t|_\psi=\partial\mathbf{x}_g/\partial t|_{\mathbf{x}}+\partial\psi/\partial t|_{\mathbf{x}}\partial\mathbf{x}/\partial\psi$, or since $\partial\mathbf{x}/\partial\psi\equiv\mathbf{e}_\psi=(\nabla\theta\times\nabla\zeta)/(\nabla\psi\cdot\nabla\theta\times\nabla\zeta)$ is the covariant base vector,

$$\left.\frac{\partial\mathbf{x}_g}{\partial t}\right|_\psi=\mathbf{v}_\parallel+\mathbf{v}_D+\left(D_\eta\Delta^+\psi-S_\psi-\bar{u}_G\frac{\partial\psi}{\partial\rho}\right)\mathbf{e}_\psi. \tag{4}$$

The three terms on the right of this equation are, respectively, zeroth, first, and second order in the small gyroradius expansion. Since $\nabla\psi\cdot\mathbf{e}_\psi=1$, this equation is also the mathematical characteristic equation for the guiding center position $\mathbf{x}_g$ that is obtained from Eqs. (1) and (2).

The guiding center equation in Eq. (4) will now be solved order by order in the small gyroradius expansion. To lowest order the particle guiding center just moves along the magnetic field $\mathbf{B}=I\nabla\zeta+\nabla\zeta\times\nabla\psi=\nabla\psi\times\nabla(q\theta-\zeta)$ with velocity $\mathbf{v}_\parallel=v_\parallel\mathbf{B}/B$, which has an oscillatory component at the transit (or bounce) frequency $\omega_t=v_\parallel\mathbf{B}\cdot\nabla\theta/B\sim v_\parallel/Rq$ because of the variation of B with θ as the particle moves along the helical path along $\mathbf{B}$ from the outboard to the inboard side of the torus. To first order the particle guiding center moves with drift velocity $\mathbf{v}_D$ mainly perpendicular to the magnetic field direction. Through zeroth and first order the particle's guiding center motion is just the normal, Hamiltonian-governed motion.[8,9] However, the particle trajectories are not necessarily governed by the Hamiltonian equations at second order because at this order the poloidal magnetic field is not an isolated, conservative magnetic field system—rather, it is a driven-dissipative system for resistive, current-carrying toroidal systems such as tokamaks.

To obtain an equation for the second order guiding center motion in a current-carrying, resistive, axisymmetric toroidal plasma, one needs to take account of the radial terms proportional to $\mathbf{e}_\psi$ in Eq. (4). To do this one first obtains the radial component of Eq. (4) from its $\nabla\psi$ component,

$$\frac{\partial\psi_g}{\partial t}=\mathbf{v}_D\cdot\nabla\psi_g+\left(D_\eta\Delta^+\psi_g-S_\psi-\bar{u}_G\frac{\partial\psi_g}{\partial\rho}\right). \tag{5}$$

Here, $\psi_g\equiv\psi_g(\rho,t)$ is the ψ-coordinate (radial) guiding center position in which $\rho\equiv\sqrt{\psi_t/\psi_t(a)}$ is the nearly stationary (except for $\bar{u}_G$) radial coordinate that is most appropriate for evolving large aspect ratio axisymmetric toroidal magnetic field systems.[2,3,6,7] The radial drift $\mathbf{v}_D\cdot\nabla\psi$ is first order, but oscillatory with the ω_t transit frequency and has a vanishing bounce average, $\langle\mathbf{v}_D\cdot\nabla\psi\rangle_{\omega_t t}=0$.[2,6] A constant of motion obtained from the first order terms in this equation is[2,6] the gyromotion-averaged canonical toroidal angular momentum for the particle guiding center, $p_{\zeta g}=mv_\parallel I/B-e\psi_g=-e\psi_g+\mathcal{O}\{\delta\}$.

For situations such as this where there is a large $\mathcal{O}\{\delta\}$ oscillatory term and higher order $\mathcal{O}\{\delta^2\}$ secular terms, the appropriate procedure to obtain the $\mathcal{O}\{\delta^2\}$ radial motion equation is to average Eq. (5) over the lower order oscillatory motion (or more formally, use the method of averaging[9]), which yields the $\mathcal{O}\{\delta^2\}$ guiding center evolution equation

$$\frac{\partial\bar{\psi}_g}{\partial t}+\bar{u}_G\frac{\partial\bar{\psi}_g}{\partial\rho}=D_\eta\Delta^+\bar{\psi}_g-S_\psi, \tag{6}$$

in which $\bar{\psi}_g=\bar{\psi}_g(\rho,t)$ is the bounce-average ψ surface the guiding center is on. This equation is valid for times longer than the inverse of the $\mathcal{O}\{\delta\}$ oscillation frequency, i.e., $t>1/\omega_t$.

The solution of Eq. (6) for the average guiding center motion $\bar{\psi}_g(\rho,t)$ will be sought using a multiple time and length scale analysis. To begin, consider the "fast, local" particle guiding center motion $\bar{\psi}_f$ of a particle immediately after a particle is located at the radial guiding center position $\rho=\rho_{g0}$,

$$\bar{\psi}_g(\rho,t=0)=\bar{\psi}_{g0}\delta(\rho-\rho_{g0}). \tag{7}$$

Since this initial condition is highly localized in radius, one can anticipate that the radial differential operator $\Delta^+\bar{\psi}_f$ will become a local slab-like operator,[2,3] $\Delta^+\bar{\psi}_f\simeq(1/\bar{a}^2)\partial^2\bar{\psi}_f/\partial x^2$ in which $\bar{a}\equiv(\langle|\nabla\rho|^2/R^2\rangle/\langle R^{-2}\rangle)^{-1/2}\sim a$ is an effective plasma minor radius[3] and $x\equiv\rho-\rho_{g0}$ is the small (dimensionless) radial distance from the initial average guiding center position ρ_{g0}. Further, writing the magnetic field diffusivity as $\bar{\nu}_e\delta_e^2$,[3] the natural scales for the time and radial length in Eq. (6) are $\Delta t=1/\bar{\nu}_e$ and $\Delta x=\bar{\delta}_e=\delta_e/\bar{a}=c/\omega_p\bar{a}$. Thus, for the fast, local response to the initial condition in Eq. (7) it is useful to define order unity, scaled time, and space variables $T\equiv\bar{\nu}_e t$ and $X\equiv x/\bar{\delta}_e=(\rho-\rho_{g0})\bar{a}/\delta_e$. In addition to the fast, local response, there will be a "slow, global" $\bar{\psi}_s$ that evolves on the macroscopic time scale $\tau_\eta \sim \bar{a}^2/6D_\eta$ of the poloidal magnetic field system. Thus, it is convenient to represent the overall average radial particle guiding center position by

$$\bar{\psi}_g(\rho,t) = \bar{\psi}_s(\rho,t) + \bar{\psi}_f(X,T). \quad (8)$$

Substituting this ansatz into Eq. (6) and using an ordering that the poloidal magnetic field is not too far from steady state so that $S_\psi \sim \bar{\nu}_e \bar{\delta}_e^2/\bar{a}^2$, one finds the following ordered equations for the fast and slow responses:

$$\mathcal{O}\{\bar{\delta}_e^0\}: \frac{\partial \bar{\psi}_f}{\partial T} + \frac{\bar{u}_G}{\bar{\nu}_e \bar{\delta}_e}\frac{\partial \bar{\psi}_f}{\partial X} = \frac{\partial^2 \bar{\psi}_f}{\partial X^2}, \quad (9)$$

$$\mathcal{O}\{\bar{\delta}_e^2\}: \frac{\partial \bar{\psi}_s}{\partial t} = \dot{\psi}. \quad (10)$$

Here, the slow, global time rate of change of the poloidal flux $\dot{\psi}$ is defined by

$$\dot{\psi}(\rho,t) \equiv D_\eta \Delta^+ \bar{\psi}_s - S_\psi - \bar{u}_G \frac{\partial \bar{\psi}_s}{\partial \rho}, \quad (11)$$

which can be nonzero for transient poloidal magnetic flux situations but vanishes for a steady-state magnetic field system. Note that the source S_ψ does not appear in the fast, local equation (9), but instead only appears in the slow, global equation (10)—because it is not important for fast, local diffusion processes but is important in the slow, global evolution of the poloidal flux.

The solution of Eq. (9) in response to the initial condition from Eq. (7) of $\bar{\psi}_g(X,T=0) = (\bar{\psi}_{g0}/\bar{\delta}_e)\delta(X)$ is $\bar{\psi}_f(X,T) = (\psi_{g0}/\bar{\delta}_e)e^{-X'^2/4T}/(4\pi T)^{1/2}$ with $X' \equiv X - (\bar{u}_G/\bar{\nu}_e\bar{\delta}_e)T$. Reverting to the regular time and space variables t and $x \equiv \rho - \rho_{g0}$ yields (for $t>0$),

$$\bar{\psi}_f(\rho,t) = \psi_{g0}\frac{e^{-(\rho-\rho_{g0}-\bar{u}_G t)^2/4\bar{D}_\eta t}}{(4\pi \bar{D}_\eta t)^{1/2}}, \quad (12)$$

in which $\bar{D}_\eta = D_\eta/\bar{a}^2 = \bar{\nu}_e \bar{\delta}_e^2$. This locally diffusive solution is valid for time scales long compared to the electron collision time $1/\bar{\nu}_e$ (so an equilibrium parallel Ohm's law is applicable[3]) but short compared to the global magnetic diffusion time scale τ_η (so the radial diffusion is locally slab-like and does not reach the cylindrical-like radial boundaries[3]),

$$1/\bar{\nu}_e \ll t \ll \tau_\eta \sim \bar{a}^2/6D_\eta. \quad (13)$$

[The condition $t > 1/\omega_t$ for the validity of Eq. (6) is not relevant here since for most tokamak plasmas $1/\omega_{te} \ll 1/\bar{\nu}_e$.] For time scales in this range, Eq. (10) for the slow, global $\bar{\psi}_s$ has the simple approximate solution

$$\bar{\psi}_s(\rho,t) \simeq \psi_0 + \int_0^t dt'\,\dot{\psi}(t') \simeq \psi_0 + t\dot{\psi}. \quad (14)$$

Here, $\psi_0(\rho)$ is the initial ($t=0$) spatial distribution of the poloidal flux.

Summing the slow and fast solutions, the overall solution for the average ψ guiding center position is (for $t>0$),

$$\bar{\psi}_g(\rho,t) = \psi_0 + t\dot{\psi} + \psi_{g0}\frac{e^{-(\rho-\rho_{g0}-\bar{u}_G t)^2/4\bar{D}_\eta t}}{(4\pi \bar{D}_\eta t)^{1/2}}. \quad (15)$$

The fast, local solution $\bar{\psi}_f$ is larger than the slow, global solution $\bar{\psi}_s$ by a factor of order $1/(\bar{\delta}_e\sqrt{\bar{\nu}_e t}) \gg 1$ for time scales satisfying Eq. (13). Note also that the mapping from ρ to ψ coordinates is unique for the $\bar{\psi}_s$ and $\bar{\psi}_f$ solutions (because $\partial \bar{\psi}_{s,f}/\partial\rho \neq 0$ for both of them) although the total solution in Eq. (15) is not (because of the asymptotic nature of the multiple scale solutions). Using the Taylor series expansion $\bar{\psi}_g = \psi_0 + (\bar{\rho}_g - \rho_{g0})\,\partial\psi/\partial\rho|_{\rho_{g0}}$ in which $\bar{\rho}_g \equiv \bar{\mathbf{x}}_g \cdot \nabla\rho$ is the average ρ position of the guiding center, this yields the same results as those given in Eqs. (13) and (14) in Ref. 2, which were derived using the method of characteristics.

The $\mathcal{O}\{\bar{\delta}_e^2\}$ diffusive spreading [with variance $\sim(\bar{D}_\eta t)^{1/2}$] and advection (with speed $\bar{u}_G$) of the average radial position of the guiding center embodied in Eqs. (12) and (15) are taken into account in the paleoclassical model by determining[2] spatial Fokker–Planck advection and diffusion coefficients $\langle \Delta \bar{x}_g\rangle/\Delta t = -\bar{u}_G$ and $\langle(\Delta\bar{x}_g)^2\rangle/\Delta t = 2\bar{D}_\eta$ from averaging $\bar{x}_g \equiv \rho - \rho_{g0}$ and $\bar{x}_g^2$ over the "probability distribution" $\bar{\psi}_g - \bar{\psi}_s = \bar{\psi}_f(\rho,t)$. Note that $\bar{\psi}_f = \bar{\psi}_g - \bar{\psi}_s$ is the thin annulus of poloidal magnetic flux that is diffusing radially[2,3] and which carries the average particle guiding center with it, because in the lower order $\mathcal{O}\{\delta\}$ analysis the constant of the motion $p_{\zeta g}$ has been identified to depend primarily on ψ_g. These effects are taken into account in the magnetic-diffusion-modified drift-kinetic equation (MDKE) (Refs. 2 and 3) by adding an $\mathcal{O}\{\bar{\delta}_e^2\}$ Fokker–Planck spatial diffusion operator $\mathcal{D}\{f\}$ to the drift-kinetic equation in Eq. (1),

$$\left.\frac{\partial f}{\partial t}\right|_\psi + (\mathbf{v}_\parallel + \mathbf{v}_D)\cdot\nabla f + \dot{\varepsilon}_g\frac{\partial f}{\partial \varepsilon_g} = \mathcal{C}\{f\} + \mathcal{D}\{f\}. \quad (16)$$

Here, the slow, global term from $(\partial\bar{\psi}_s/\partial t|_{\mathbf{x}})(\partial f/\partial\psi)$ on the left of Eq. (1) has been included in $\mathcal{D}\{f\}$ in order to incorporate all the $\mathcal{O}\{\bar{\delta}_e^2\}$ magnetic field diffusion effects in one operator. While the Fokker–Planck coefficients were derived in the time interval indicated in Eq. (13), the MDKE is valid for all time scales longer than the effective electron collision time (i.e., $t > 1/\bar{\nu}_e$) because the full $\mathcal{D}\{f\}$ operator in Ref. 2 preserves the correct geometrical properties of the diffusion process. The flux surface average of this effective Fokker–Planck spatial diffusion operator is[2]

$$\langle \mathcal{D}\{f(\rho)\}\rangle \simeq -\dot{\psi}\frac{\partial f}{\partial\psi} + \frac{1}{V'}\frac{\partial}{\partial\rho}\left[V'\bar{u}_G f + \frac{\partial}{\partial\rho}(V'\bar{D}_\eta f)\right], \quad (17)$$

in which inverse aspect ratio squared terms have been neglected[3] and $V' \equiv dV/d\rho$, where $V(\rho)$ is the volume of the ρ flux surface. The velocity-space moments of the MDKE yield[2] the proper fluid moment equations[6,7] for density and temperature in which the transport fluxes are measured relative to the ψ surfaces, taking account of the $\bar{u}_G$ grid speed and $\dot{\psi}$ transient effects, but now include the additional guiding center radial diffusion effects due to the magnetic field diffusivity D_η in Eq. (17).

In summary, with this background in mind, specific responses to the two conclusions at the end of the comment[1] are:

(i) The underlying concept of paleoclassical transport appears to be based on a misinterpretation of resistive dissipation of poloidal flux as a radial random walk process of field lines (flux bundles).

The paleoclassical concept does not involve a random walk process. Rather, as indicated above and in Refs. 2 and 4, the key hypothesis of the paleoclassical model results from transforming (mapping) the guiding center particle trajectory equations and the drift-kinetic equation from laboratory to poloidal flux coordinates in situations where the poloidal magnetic flux obeys a slowly evolving diffusion equation as in Eq. (2) and following through all the $\mathcal{O}\{\bar{\delta}_e^2\}$ consequences of this transformation. Physically, in transforming to poloidal flux coordinates in which to perform further kinetic analysis on and relative to the ψ surfaces, one is effectively transforming to a locally slowly diffusing ψ coordinate system relative to and about which the particle guiding center is rapidly oscillating. Mathematically, in mapping to poloidal flux coordinates one is introducing slowly evolving $\mathcal{O}\{\bar{\delta}_e^2\}$ parabolic (diffusive) characteristics into the drift-kinetic and particle guiding center equations.

(ii) The mathematical manipulations leading to the conversion of an advection term in the drift Fokker–Planck equation into a diffusion term, appear to lack a systematic basis.

Equations (4)–(15) above have presented a systematic ordering of the particle guiding center motion to lowest order (parallel streaming), first order (oscillatory radial drift about a ψ surface leading to the constant of the motion $p_{\zeta g} \simeq -e\psi_g$), and second order (advection and diffusion of the average guiding center position $\bar{\psi}_g$) in the small gyroradius expansion. The conversion of an initially advective term into a diffusion term occurs because to lowest order the distribution function f is a function of ψ, or more precisely the canonical toroidal angular momentum $p_{\zeta g} = m v_\parallel I/B - e\psi_g \simeq -e\psi_g + \mathcal{O}\{\delta\}$. Then, since at $\mathcal{O}\{\bar{\delta}_e^2\}$ the average radial position $\bar{\psi}_g$ of the particle's guiding center diffuses as indicated in Eq. (12), so do the bounce-average canonical toroidal angular momentum $\bar{p}_{\zeta g} \simeq -e\bar{\psi}_f$ and distribution function $f(\bar{p}_{\zeta g})$.

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