Response to "Comment on 'Derivation of paleoclassical key hypothesis' " [Phys. Plasmas 15, 014703 (2008)]

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The comment by Thyagaraja *et al.*[1] sets forth its own view of how all plasma transport is determined but does not reflect the physical or mathematical content of the paleoclassical model. The paleoclassical model is derived[2] from a treatment of effects that result from a combination of the drift-kinetic equation, Faraday's law, and Ohm's law. It does not contradict electrodynamics; rather, it incorporates the electrodynamic consequences of Faraday's law plus a resistive Ohm's law in determining the slow, diffusive evolution of the poloidal magnetic flux and its effects on particle guiding center trajectories and on the drift-kinetic equation when they are transformed (mapped) to poloidal flux surfaces.

The key hypothesis of the paleoclassical model of radial electron heat transport is that[3] electron guiding centers diffuse radially along with thin annuli of poloidal magnetic flux in resistive, current-carrying axisymmetric toroidal plasmas, e.g., tokamak plasmas. Its derivation[2] is based on transforming (mapping) the drift-kinetic equation from laboratory to poloidal magnetic flux coordinates in situations where the poloidal flux ψ obeys a slowly evolving (compared to the particle transit or bounce frequency) diffusion equation, determining the mathematical characteristic curves (effective particle trajectories) including the slow poloidal flux diffusion effects, and taking account of the resultant slowly radially diffusing particle trajectories via a Fokker–Planck spatial diffusion operator. A simpler model that illustrates the key points involved in the derivation was presented in Ref. 4. The salient points are:[4] "(1) particles oscillate rapidly about a ψ surface; (2) ψ obeys a slowly evolving diffusion equation; and (3) further analysis of the kinetic equation is to be carried out on and relative to ψ surfaces"—all of which are satisfied by the derivation in Ref. 2.

Finally, most relevant to the comment by Thyagaraja *et al.*,[1] a direct derivation of the radial diffusion of the particle guiding center trajectories from the "Lorentz–Newton" equations $m d\mathbf{v}/dt = e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$ has been presented.[5] The first step in the derivation is to expand the Lorentz–Newton equations in a small gyroradius expansion and gyroaverage using standard procedures to obtain the usual guiding center motion equation

$$\partial \mathbf{x}_g/\partial t\big|_{\mathbf{x}} = \mathbf{v}_\parallel + \mathbf{v}_D. \tag{1}$$

After that, the key point is to realize that in the usual guiding center motion equation the time derivative of the guiding center position $\mathbf{x}_g$ is taken at a constant laboratory position $\mathbf{x}$. However, drift-kinetic analyses for neoclassical[6] and microturbulence-induced transport are carried out in poloidal magnetic flux coordinates and the "radial" transport fluxes are determined relative to poloidal flux surfaces. The evolution equation for the poloidal magnetic flux ψ deduced from a combination of a component of Faraday's law and a parallel Ohm's law is a diffusion equation for ψ,[3,6,7]

$$\left.\frac{\partial\psi}{\partial t}\right|_{\mathbf{x}} + \bar{u}_G \frac{\partial\psi}{\partial\rho} = D_\eta \Delta^+ \psi - S_\psi. \tag{2}$$

All the terms in this ψ evolution equation are second order in the small gyroradius expansion (i.e., $\mathcal{O}\{\delta^2\}$ in which $\delta \sim \varrho/L_\perp$) since the magnetic field diffusion coefficient is $D_\eta \equiv \eta_\parallel^{\rm nc}/\mu_0 \sim \nu_e (c/\omega_p)^2$ and the electromagnetic skin depth $\delta_e \equiv c/\omega_p$ is of order the gyroradius $\varrho \equiv v_\perp/\omega_c = m v_\perp/eB$,

Since drift-kinetic analyses for neoclassical and microturbulence-induced plasma transport are carried out on ψ surfaces, the particle guiding center equation needs to be transformed from laboratory to the poloidal magnetic flux coordinates ψ,θ,ζ. Taking the time derivative of the guiding center position $\mathbf{x}_g$ at constant ψ using $\partial\mathbf{x}_g/\partial t|_{\mathbf{x}} = \partial\mathbf{x}_g/\partial t|_\psi + \partial\psi/\partial t|_{\mathbf{x}} \partial\mathbf{x}/\partial\psi$ and Eq. (2) yields

$$\left.\frac{\partial\mathbf{x}_g}{\partial t}\right|_\psi = \mathbf{v}_\parallel + \mathbf{v}_D + \left(D_\eta\Delta^+\psi - S_\psi - \bar{u}_G\frac{\partial\psi}{\partial\rho}\right)\mathbf{e}_\psi, \tag{3}$$

in which $\mathbf{e}_\psi \equiv \partial\mathbf{x}/\partial\psi$ is the covariant base vector in the ψ direction ($\mathbf{e}_\psi\cdot\nabla\psi = 1$).

A systematic procedure for solving this guiding center equation order by order in the small gyroradius expansion has been presented in Ref. 5. The analysis there shows that the particle guiding center motion: (0) to lowest order is just parallel streaming at velocity $\mathbf{v}_\parallel$ along the $\mathbf{B}$ field; (1) to first order yields an oscillatory radial drift $\mathbf{v}_D\cdot\nabla\psi$ about a ψ surface leading to the canonical toroidal angular momentum constant of guiding center motion $p_{\zeta g} = m v_\parallel I/B - e\psi_g \simeq -e\psi_g + \mathcal{O}\{\delta\}$; and finally (2) to second order produces radial advection (at speed $\bar{u}_G$) and diffusion (with coefficient D_η) of the bounce-average guiding center position $\bar{\psi}_g$. The average particle guiding center canonical toroidal angular momentum $\bar{p}_{\zeta g} \simeq -e\bar{\psi}_g$ also diffuses radially at second order in the small gyroradius expansion. The particle trajectories are not Hamiltonian at second order because at this order the poloidal magnetic field is not an isolated, conservative magnetic field system but instead is a driven-dissipative system for resistive, current-carrying toroidal systems such as tokamaks. In transforming to poloidal flux coordinates in which

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to perform further kinetic analysis on and relative to the ψ surfaces, one is effectively transforming to a locally slowly diffusing ψ coordinate system relative to and about which the particle guiding center is rapidly oscillating.

With this background in mind, specific responses to the primary assertions in the comment by Thyagaraja *et al.*[1] are:

The paleoclassical hypothesis (PCH) assumes that Eq. (2) describes the Brownian motion of poloidal flux, even at steady state. The paleoclassical model does not invoke Brownian motion as the physical process underlying the diffusion equation for ψ in Eq. (2). Apparently the confusion arises from statements in the original papers attempting to motivate a phenomenological basis for the paleoclassical model:[8] "Magnetic field diffusion…is induced by the plasma resistivity η. It causes magnetic field lines to diffuse perpendicular to **B** with a diffusion coefficient $D_\eta \simeq \eta_0 \equiv \nu_e (c/\omega_p)^2$, which implies a diffusive radial step $\Delta x \simeq \delta_e \equiv c/\omega_p \cdots$ in a collision time $\Delta t \simeq 1/\nu_e$." As derived explicitly in Refs. 3, 6, and 7 and discussed above, the ψ evolution equation comes from a combination of Faraday's law and a parallel Ohm's law. That is, ψ obeys a diffusion equation but does not result from a Brownian motion process.

The poloidal flux ψ has no relevance to individual charged particle motions, it is merely a convenient way of representing **B**. However, because of toroidal axisymmetry the canonical toroidal angular momentum $p_{\zeta g}$ constant of guiding center motion through $\mathcal{O}\{\delta\}$ depends explicitly and predominantly on ψ. Also, when one transforms from laboratory to poloidal flux coordinates, ψ becomes the radial coordinate on which and relative to which subsequent drift-kinetic analyses are performed for neoclassical and microturbulence-induced transport. Thus, the poloidal flux ψ (or more precisely the poloidal flux minus some additive constant, hence a thin annulus of poloidal flux) becomes critically important for theoretical analyses of plasma transport processes in tokamaks.

PCH can only generate additional diffusion of electron guiding centres by adding new terms to the Lorentz force equation. This is true. However, as explicitly derived in Eqs. (5)–(13) in Ref. 2, Eqs. (4)–(15) of Ref. 5 and discussed above, the relevant terms come about naturally by transforming the equations of motion from laboratory (**x**) to poloidal flux (ψ) coordinates for situations where ψ obeys a diffusion equation.

The Lorentz force equations are the exact characteristics of the hyperbolic Vlasov equation. It follows that any "drift kinetic equation" derived from averaging over Larmor gyrations…cannot contradict the predictions of the Lorentz–Newton and the equivalent Vlasov equations. This is true. But the totally collisionless, dissipation-free Vlasov equation is irrelevant here. Rather, in order to obtain the poloidal flux evolution equation in Eq. (2), one needs a parallel Ohm's law which results from including Coulomb collision effects in a plasma. Then, as indicated in Eq. (3) above and in more detail in Ref. 2, the diffusive term in Eq. (2) introduces parabolic (diffusive) characteristics into the relevant drift-kinetic and particle guiding center equations, at second order in the small gyroradius expansion. Also, Coulomb collision effects are needed in the drift-kinetic equation since they are what cause classical and neoclassical transport of plasma across ψ surfaces.

According to both classical and neoclassical kinetics the particle diffusivity must scale like $1/B^2$ ("gyro-Bohm") whilst the magnetic field diffusivity is independent of B. Clearly the B-independent transport rates by PCH violate the fundamental ordering, and would predict finite particle transport even with very large fields! The implication of this comment is that all collision-based diffusion coefficients derived from a plasma kinetic equation that includes a Coulomb collision operator and the Maxwell equations must scale inversely with magnetic field strength B and, in particular, that none can scale independently of B. However, the paleoclassical scaling of particle diffusivity with the plasma resistivity is certainly consistent with both dimensional analysis[9] and scale invariance transformations[10] (for either the collisional high-β or resistive MHD scaling laws[10]). In fact, Kadomtsev argues[11] that c/ω_p is the most relevant dimensional scale length for determining the "neo-Alcator scaling" of energy confinement in Ohmic-level plasmas. Detailed comments on the relevant dimensionless variables for paleoclassical transport have been discussed in paragraphs around Eqs. (63) and (64) in Ref. 12.

Our principal conclusion is that the paleoclassical hypothesis contradicts both the Lorentz force equation and direct asymptotic solutions of the generally accepted equations for the charged particle distribution functions in the classical and neoclassical regimes.[13] As explained in going from Eq. (1) to Eq. (3) above and explicitly in Ref. 5, the paleoclassical hypothesis does not violate the Lorentz force equation; rather, it takes account of the electrodynamic effects of transforming the Lorentz force equation from its time derivative being taken at a constant laboratory position **x** to being taken at constant ψ in situations where the poloidal flux obeys a diffusion equation as given in Eq. (2). The resultant paleoclassical effects on the drift kinetic equation are second order in the small gyroradius expansion. The reasons why the derivation of the drift kinetic equation in Ref. 13 is incomplete regarding the second order paleoclassical effects and terms are explained in the paragraphs after Eqs. (5) and (19) in Ref. 2 and in greater detail in the fourth and fifth paragraphs of Ref. 5.

While electron heat transport has been found in some experiments to lie close to the paleoclassical level,[12] *this in no way validates the model's physical basis. Indeed any microturbulence model implying a step length of $\delta_e = c/\omega_p \cdots$ and a turbulent decorrelation rate of the order of ν_e, would lead to similar numerical values for the "effective turbulent diffusivity": $\nu_e \delta_e^2 \simeq \eta/\mu_0$*. The paleoclassical model of radial electron heat transport[3] involves far more than a scaling with $D_\eta = \eta/\mu_0 \sim \nu_e \delta_e^2$—a helical multiplier $M \sim 10 \gg 1$ which is critical in "collisionless" plasmas, a collisional "Alcator-scaling" regime, reduced transport around low order rational surfaces, etc. In addition to the initial[8,12] "back-of-the-envelope" comparisons of the paleoclassical model predictions with electron heat diffusivities from experiments, a much more extensive modeling paper which performs 18 tests in seven devices has recently been published.[14] While the rather favorable and extensive agreement found[14] with mostly Ohmic-level tokamak plasmas (to within a factor ~ 2) does not "validate" the theoretical model, it does indicate that tokamak plasmas seem to understand the paleoclassical model predictions and encourage further development of it, or perhaps other models that could produce the various paleoclassical regimes[3] and such favorable experimental comparisons.[14]

…a challenging observation for PCH is posed by runaway electron populations which clearly do not experience paleoclassical transport and have coexisted with cold resistive plasmas in JET for up to $\sim\mathcal{O}(100)$ *resistive diffusion times.*[15] Since the cited paper makes no such statement, this assertion is apparently an interpretation by Thyagaraja *et al.*[1] The relevant experimental results presented in Ref. 15 occur in the aftermath of a major disruption in the JET where it is observed that runaways up to 35 MeV are produced in $\sim 10-15$ ms. Ultimately, "the loop voltage drops to nearly zero and the runaway current becomes a substantial fraction of the original plasma current."[15] Not enough details are provided to characterize either the fraction of the current (~ 1 MA) carried by runaways or the background "cold plasma" to determine the possible role of paleoclassical transport. However, the basic paleoclassical model must be modified[2] for these plasmas in which most of the current is carried by collisionless runaway electrons that suffer very little Coulomb collisional drag since the near zero loop voltage implies the Ohmic transformer is not resistively driving any substantial current in these postdisruption, runaway-electron-dominated plasmas. Paleoclassical transport is induced by[2,3] $\eta\langle\mathbf{J}\cdot\mathbf{B}\rangle \propto (\eta/\mu_0)\Delta^{+}\psi$, which in the absence of noninductive current sources is proportional to the loop voltage. When the poloidal magnetic flux is determined both by a "vacuum" (V) current (ψ_V, in this case from the plasma-decoupled runaway current) and some current driven inductively via the plasma resistivity (ψ_J), the relevant diffusion coefficient[2] for paleoclassical transport becomes smaller than D_η by a factor of $\psi'_J/(\psi'_V+\psi'_J)$. This factor is apparently quite small for the JET runaway-dominated postdisruption plasmas,[15] which is qualitatively consistent with the long confinement of runaway electrons there. The paleoclassical predictions for radial diffusion of runaway electrons in normal tokamak plasmas, which compare favorably with experimental inferences, are commented on at the end of the second paragraph of Ref. 12.

In summary, the comment by Thyagaraja *et al.*[1] provides its own interpretation of many plasma processes and transport issues. However, it incorrectly depicts the paleoclassical model and does not actually address the physical processes or mathematics in the paper[2] on which it purports to provide a comment. In particular, the paleoclassical model does not contradict electrodynamics; rather, it takes account of the previously neglected second order effects (in the small gyroradius expansion) of the electrodynamic ψ evolution equation in Eq. (2), which results from Faraday's law and a parallel Ohm's law, on the particle guiding center trajectories and hence on the drift-kinetic equation for a resistive, current-carrying axisymmetric toroidal plasma such as a tokamak.

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