Transport equations in tokamak plasmas\textsuperscript{a)}

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(Received 20 November 2009; accepted 29 December 2009; published online 8 April 2010)

Tokamak plasma transport equations are usually obtained by flux surface averaging the collisional Braginskii equations. However, tokamak plasmas are not in collisional regimes. Also, ad hoc terms are added for neoclassical effects on the parallel Ohm’s law, fluctuation-induced transport, heating, current-drive and flow sources and sinks, small magnetic field nonaxisymmetries, magnetic field transients, etc. A set of self-consistent second order in gyroradius fluid-moment-based transport equations for nearly axisymmetric tokamak plasmas has been developed using a kinetic-based approach. The derivation uses neoclassical-based parallel viscous force closures, and includes all the effects noted above. Plasma processes on successive time scales and constraints they impose are considered sequentially: compressional Alfvén waves (Grad–Shafranov equilibrium, ion radial force balance), sound waves (pressure constant along field lines, incompressible flows within a flux surface), and collisions (electrons, parallel Ohm’s law; ions, damping of poloidal flow). Radial particle fluxes are driven by the many second order in gyroradius toroidal angular torques on a plasma species: seven ambipolar collision-based ones (classical, neoclassical, etc.) and eight nonambipolar ones (fluctuation-induced, polarization flows from toroidal rotation transients, etc.). The plasma toroidal rotation equation results from setting to zero the net radial current induced by the nonambipolar fluxes. The radial particle flux consists of the collision-based intrinsically ambipolar fluxes plus the nonambipolar fluxes evaluated at the ambipolarity-enforcing toroidal plasma rotation (radial electric field). The energy transport equations do not involve an ambipolar constraint and hence are more directly obtained. The “mean field” effects of microturbulence on the parallel Ohm’s law, poloidal ion flow, particle fluxes, and toroidal momentum and energy transport are all included self-consistently. The final comprehensive equations describe radial transport of plasma toroidal rotation, and poloidal and toroidal magnetic fluxes, as well as the usual particle and energy transport. © 2010 American Institute of Physics. [doi:10.1063/1.3335486]

I. INTRODUCTION

Transport equations for describing tokamak plasmas were initially advanced in the late 1970s. They were developed by adapting the collisional Braginskii\textsuperscript{1} transport equations for plasma density and energy transport to the tokamak toroidal geometry. In addition, many other terms have been added over the years in an ad hoc manner to include other processes important in tokamak plasmas: neoclassical effects on the parallel Ohm’s law\textsuperscript{2,3} [trapped particle effects on the resistivity, bootstrap (bs) current], “anomalous” plasma transport induced by microturbulence, auxiliary heating, current-drive (CD) and flow sources and sinks, effects of small magnetic field departures from axisymmetry, etc. The resultant transport equations form the basis for ONETWO,\textsuperscript{4} TRANSP,\textsuperscript{5} and other transport modeling codes that are used to model present and future tokamak plasma experiments.

However, most tokamak plasmas are not in a collisional regime—collision lengths are typically much longer than the toroidal circumference of the tokamak. More rigorous and self-consistent plasma transport equations need to be developed from a kinetic-based approach. Also all the effects indicated in the preceding paragraph need to be included naturally and self-consistently.

In this paper we develop\textsuperscript{6,7} self-consistent radial plasma transport equations from velocity-space (fluid) moments of a kinetic-based approach that includes all these effects for nearly axisymmetric tokamak plasmas using neoclassical-based parallel viscous force closures.\textsuperscript{3} This new approach accounts for lower order radial force balance constraints and solves for the flows within flux surfaces before determining the net radial transport equations. This procedure is analogous to that used for determining radial neoclassical plasma transport equations in nonaxisymmetric stellarator plasmas—see for example Ref. 8 and references cited therein. This new approach was originally developed\textsuperscript{6,7} for determining the toroidal rotation in tokamak plasmas. This paper extends that analysis to include heat transport equations and takes into account the poloidal heat flows that were neglected there.

For additional details see Refs. 6 and 7.

This paper is organized as follows. Section II discusses the fundamental kinetic equation from which the analysis begins and the resultant fluid moment equations that are used. Also, the key assumptions, small gyroradius expansion used for ordering various terms and lowest order radial force balance considerations are presented there. Section III describes the determination of the first order plasma flows within flux surfaces. The transient evolution of poloidal and
toroidal magnetic fluxes and their effects on plasma transport are described in Sec. IV. The radial particle fluxes induced by the many toroidal torques on the plasma\footnote{7} are summarized in Sec. V. Also, the toroidal plasma rotation equation for a tokamak\footnote{6} obtained by setting the net radial current induced by the nonambipolar fluxes to zero to enforce plasma quasineutrality is summarized there. This key equation also determines the radial electric field and the resultant ambipolar radial particle flux. Section VI develops the electron and ion energy transport equations. Finally, the results of this paper are summarized in Sec. VII which also discusses some key new features of this comprehensive transport model.

II. BASIC EQUATIONS AND ASSUMPTIONS

We begin with a plasma kinetic equation (PKE) that includes the Fokker–Planck Coulomb collision operator $C(f)$ and kinetic sources $S(f)$ for each plasma species,

$$\frac{df}{dt} + \mathbf{v} \cdot \nabla f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{df}{dv} = C(f) + S(f).$$

(1)

Fluid moment equations for the species density $n$, flow velocity $\mathbf{V}$, temperature $T$, and conductive heat flow $q$ are obtained by taking velocity-space moments $\int d^3v [1, m \mathbf{v}, m^2 \mathbf{v}/2, \mathbf{v}(m^2 \mathbf{v}/2)]$ of this PKE,

$$\frac{\partial n}{\partial t} + \mathbf{v} \cdot \nabla n = S_n,$$

(2)

$$\frac{\partial}{\partial t} (mn \mathbf{V}) + \nabla \cdot (mn \mathbf{VV}) = nq(\mathbf{E} + \mathbf{V} \times \mathbf{B}) - \nabla p - \nabla \cdot \mathbf{\Pi} + \mathbf{R}_V + S_V,$$

(3)

$$\frac{3}{2} \frac{\partial p}{\partial t} + \nabla \cdot \left( q + \frac{5}{2} p \mathbf{V} \right) = Q + \mathbf{V} \cdot \nabla p - \mathbf{\Pi} \cdot \nabla \mathbf{V} + S_E,$$

(4)

$$\frac{\partial q}{\partial t} = \frac{q - m}{m} \mathbf{V} \times \mathbf{B} - \frac{T}{m} \left( \frac{5}{2} \nabla T + \nabla \cdot \mathbf{\Theta} - \mathbf{R}_q \right) - \nabla \left( \mathbf{V} q + q \mathbf{V} + \frac{2}{3} \mathbf{V} \cdot \mathbf{q} \mathbf{I} \right) + \frac{5}{2} p \mathbf{V} \cdot \nabla \mathbf{V} + S_q^1 + \cdots.$$  

(5)

As indicated, in this conservative form of these equations the fluid moment properties are evaluated at a laboratory position $\mathbf{x}$. Here, the species pressure is $p = nT$, and $\mathbf{R}_V$ and $\mathbf{R}_q$ are the collisional friction and heat friction forces. The density, momentum, net energy, and net heat flow sources are $S_n$, $S_V$, $S_q^1 = \mathbf{S}_q - \mathbf{V} \cdot \mathbf{S}_V - (m^2 \mathbf{V}/2) S_T$, and $S_q^2 = S_q - (5T/2m) \times (\mathbf{V} \cdot \mathbf{S}_V - m \mathbf{V} S_T)$. The $\cdots$ at the end of Eq. (5) indicates higher order terms that will be neglected here. The remaining notation and definitions are relatively standard.\footnote{1-3} The fluid moments required to close these equations are the stress tensor $\mathbf{\Pi}$ and heat stress tensor $\mathbf{\Theta}$; the needed parallel viscous forces they induce are given in Sec. III.

A number of assumptions are made to facilitate the analysis. (1) Particle gyroradii are small which to zeroth order yields magnetohydrodynamic (MHD) force balance equilibrium and the radial ion force balance, flow on flux surfaces at first order and second order “radial” transport fluxes. (2) Magnetic flux surfaces are nested and toroidally axisymmetric to lowest order (i.e., there are no magnetic islands in the region of interest). (3) Nonaxisymmetries (NA) in the magnetic field are first order in the gyroradius, which causes the toroidal flow damping rate to be one order smaller than the poloidal flow damping rate. (4) Collision lengths are long compared to the plasma toroidal circumference so plasma properties are constant on magnetic flux surfaces, which is valid for most tokamaks except possibly in a cold plasma edge. (5) Electron and ion distribution functions are Maxwellian to lowest order in the gyroradius expansion (e.g., there are no radio-frequency-wave-induced zeroth order distribution distortions). (6) Microinstability-induced plasma fluctuations are gyroradius small and lead mostly to second order “anomalous” plasma transport across flux surfaces. (7) Impurity flow velocities are approximately equal to hydrogenic ion velocities, which simplifies inclusion of impurity effects. Finally, (8) magnetic field transients are slow enough that they only occur on the plasma transport or longer time scale.

A small gyroradius expansion is used to order the various terms in the fluid moment equations: $\delta = \sqrt{a/\kappa_0} \ll 1$ in which $\kappa$ is the particle gyroradius in the magnetic field and $a$ is the minor radius of the plasma. For example, the plasma pressure is expanded as

$$p(\mathbf{x}) = p_0(\rho) + \delta [\tilde{p}_1(\rho, \theta) + \tilde{p}_2(\rho, \theta, \zeta)] + \mathcal{O}(\delta^2).$$

(6)

The lowest order species pressure $p_0$ is only a function of the magnetic flux surface label $\rho$. The first order toroidal-angle-averaged pressure $\tilde{p}_1(\rho, \theta)$ represents the Pfirsch–Schluter poloidal ($\theta$) pressure variation on a flux surface. Finally, $\tilde{p}_2(\rho, \theta, \zeta)$ represents the gyroradius-small nonaxisymmetric perturbations induced by externally imposed magnetic fields or collective plasma instabilities.

Because of lowest order axisymmetry in the toroidal angle $\zeta$, perturbations will be expanded in a Fourier series: $\tilde{p}_1 = \sum \tilde{p}_1 e^{-in\zeta}$, $\tilde{p}_2 = (1/2\pi)^{1/2} \int_0^{2\pi} d\zeta e^{in\zeta} \tilde{p}_2$. The $n=0$ term defines the average over toroidal angle,

$$p(\mathbf{x}) = \frac{1}{2\pi} \int_0^{2\pi} d\zeta p(x) = p_0(\rho) + \delta \tilde{p}_1(\rho, \theta) + \mathcal{O}(\delta^2).$$

(7)

Perpendicular scale lengths of fluctuations will be presumed to be on the gyroradius scale. Hence, perpendicular derivatives of fluctuations will be assumed to be large: $\nabla_\perp \tilde{p}_1 \sim (1/\delta) \delta \sim \delta$. However, parallel derivatives of fluctuations will be assumed to be on the macroscopic ($\sim a$) scale and hence small: $\nabla \parallel \tilde{p}_1 \sim \delta \sim \delta$. Derivatives of equilibrium components $p_0, \tilde{p}_1$ will be $\mathcal{O}(\delta^2)$. The magnetic field will be represented by an average $\mathbf{B}_0 = \mathbf{B}_0 \parallel + \mathbf{B}_0 \perp \mathbf{\xi} \times \nabla \psi_p = \nabla \psi_p \times (\psi_p \mathbf{\perp} - \psi_p \mathbf{\parallel}) + \mathbf{\xi} \times (\psi_p \mathbf{\perp} \mathbf{\parallel} - \psi_p \mathbf{\perp} \mathbf{\parallel})$ plus small $\mathcal{O}(\delta)$ perturbations $\mathbf{B}$ due to three-dimensional (3D) NA and collective responses: $\mathbf{B} = \mathbf{B}_0(\rho, \theta) + \delta \mathbf{B}_1 + \mathcal{O}(\delta^2)$. Here,
\[ I(\psi_p) = RB_{\psi}, \quad \psi_p, \psi_t \text{ are the poloidal and toroidal magnetic fluxes.} \]

The electric field will be represented as a sum of scalar and vector potentials: \( E = -\nabla \phi + E^A \), in which the inductive electric field is \( E^A = -\sigma \nabla/\sigma_t \). Since the \( \xi \)-average vector potential is \( A = \psi_t \nabla \theta - \psi_p \nabla \xi \), the \( \xi \)-average toroidal inductive electric field will be written as \( E_\xi^A = (\vartheta \nabla/\vartheta_t + \psi_A) \nabla \xi - \psi_t \nabla \theta - \nabla \{(\nabla \theta \cdot \nabla \xi) \} \), in which \( 2 \pi \sigma \nabla/\sigma_t = \nabla \theta \) is the poloidal loop voltage and the dots over the magnetic fluxes indicate their partial time derivatives at a given laboratory position \( x \).

The lowest order (\( \delta \)) fluid moment equations in Eqs. (2)–(4) yield a two-fluid model of the ideal MHD model. Compressional Alfvén waves perpendicular to the magnetic field enforce \( J_0 \times B_0 = 0 = \nabla P \) on the fast Alfvén time scale. This relation in combination with a two-fluid Ohm’s law \( \mathbf{E}_0 + \mathbf{V} \times \mathbf{B}_0 = (J_0 \times B_0) / \mathcal{N}_e \) yields the equilibrium (\( T > 1/\nu_e \), neglecting electron inertia) radial force balance \( 0 = e_p \{ n_e (E + \mathbf{V} \times \mathbf{B}) - \nabla p_i \} \), which in our toroidal coordinates yield \( \Omega = \mathbf{V} \cdot \nabla \xi = -\left( \frac{d \phi}{dp_i} + \frac{1}{n_i q_i} \nabla e \cdot \nabla \psi_t \right) \).

In cylindrical-type coordinates \( d \psi_p = B_p \delta \rho d \rho \) this relation between toroidal and poloidal flows, the radial electric field and radial ion pressure gradient for each ion species is \( \mathbf{V}_i = E_i / B_\rho \left[ \frac{1}{n_e (q_i)} \right] (d p_i / d \rho) + (q_i / B_\rho) \mathbf{V}_e \).

Maxwellianization of the electron, ion distributions on their collision times of \( 1/\nu_e, 1/\nu_i \) cause \( n, T \) to be constant over collision lengths \( \lambda_e, \lambda_i \) and hence on flux surfaces. Also, species flow velocities \( \mathbf{V}_i \) become physically meaningful on the collision time scales. As discussed in Sec. III, to first order \( \delta \) in the small gyroradius expansion the plasma flows on flux surfaces. The radial flows across flux surfaces are second order \( \delta^2 \).

### III. CURRENTS, FLOWS ON FLUX SURFACES

This work extends our previous analyses of flows \( \delta \) to include heat flow effects. At order \( \delta \) the flows are on magnetic flux surfaces and can be represented by components in the \( \theta, \xi \) or \( \parallel, \wedge \) directions. \( \delta \)

\[ \mathbf{V}_1 = e_p \nabla \cdot \nabla \theta + e_\xi \nabla \cdot \nabla \xi = \nabla \cdot \mathbf{B}_0 / B_0 + \nabla \lambda \cdot \nabla \xi. \]

The conductive heat flow is represented similarly. The cross flow in Eq. (9) indicates \( E_0 \times B_0 \) and diamagnetic flows,

\[ \nabla \lambda \cdot \nabla \theta = B_0 / B_0 \left( \frac{d \phi_0}{dp_i} + \frac{1}{n_i q_i} \nabla e \cdot \nabla \psi_t \right). \]

To lowest order equation (5) yields the diamagnetic cross heat flow \( \mathbf{q}_\lambda = (n_0 B_0 T_0 / q_0) (B_0 / \nabla \psi_t / B_0^2) (d T_0 / d \psi_t) \).

On transport time scales the first order flows and heat flows are incompressible due to sound wave equilibration along field lines. Then, the poloidal flow function \( U_0 \) becomes constant on a flux surface and is given by \( \delta \)

\[ U_0(\psi_p) = \frac{\mathbf{V}_1 \cdot \nabla \theta}{B_0^2} = \frac{B_0 \cdot \nabla \theta}{B_0^2} + \frac{\nabla \lambda \cdot \nabla \theta}{B_0^2}. \]

The poloidal heat flow function is similarly represented: \( Q_s(\psi_p) = \nabla \cdot \mathbf{q}_s / B_0^2 \cdot \mathbf{V}_0 \cdot \nabla \theta = (5 / 2) \times ( \langle n_0 B_0 T_0 / q_0 \rangle ) (d T_0 / d \psi_t) \). Thus, we obtain

\[ Q_s(\psi_p) = \frac{-2}{5 n_0 T_0 q_0 B_0^2} \mathbf{B}_0 \cdot \mathbf{q}_s - \frac{1}{q_0 B_0^2} \frac{d T_0}{d \psi_t}. \]

Multiplying Eq. (11) by \( n_0 q_0 B_0^2 \), summing over species and flux surface averaging (ASA) yields \( \langle \delta \rangle \mathbf{K}_\delta = \langle \delta \rangle \mathbf{J}_\delta \).

\[ \frac{\mathbf{K}_\delta}{B_0^2} = \mathbf{J}_\delta - \frac{1}{2} \frac{d \rho_0}{d \psi_t} \left( 1 - \frac{B_0^2}{B_0^2} \right). \]

The first term represents the ASA parallel current contribution while the second indicates the Pfirsch–Schlüter current. In tokamaks the ASA parallel current is defined in terms of the poloidal flux \( \psi_p \) by \( \delta \)

\[ \mu_0 \mathbf{B}_0 \cdot \mathbf{J}_\delta = I \langle R^{-2} \rangle \Delta^* \psi_p, \]

in which the second order cylindrical-type operator is

\[ \Delta^* \psi_p = \frac{I}{\langle R^{-2} \rangle} \frac{\partial}{\partial \rho} \left( \frac{[\mathbf{V}_{\rho} (\nabla \theta)]^2}{R^2} \right) \frac{1}{\vartheta} \frac{\partial \psi_t}{\partial \rho}. \]

Here and hereafter \( V = dV / d \rho \) is the radial derivative of the volume \( V(\rho) = \int_0^\rho d \chi \) of the \( \rho \) flux surface.

Flows on flux surfaces in tokamak plasmas are obtained by solving coupled momentum and heat flow equations for electrons and hydrodynamic and impurity ions. For example, they can be evaluated using the NCLASS code. However, assuming, as is often the case, impurities are in the plateau or Pfirsch–Schlüter collisionality regimes, the flow velocities of impurities are approximately equal to those of the hydrogenic ions and an analytic analysis is possible. Making this assumption facilitates the straightforward inclusion of impurities, noninductive sources, dynamic, and other current and flow drives in the parallel Ohm’s law and poloidal ion flow equations.

The parallel Ohm’s law is obtained from the equilibrium \( \mathbf{J}_\parallel = \mathbf{J}_0 \) first order FSA parallel electron momentum equation (3) and heat flow equation (5), \( \delta \)

\[ \begin{bmatrix} \frac{d}{dt} \mathbf{B}_0 \cdot \mathbf{V}_{\parallel, e} \\ \frac{d}{dt} \mathbf{B}_0 \cdot \mathbf{V}_{\parallel, i} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_0 \cdot \mathbf{V}_{\parallel, e} + \mathbf{B}_0 \cdot \mathbf{R}_{\chi e} + \mathbf{B}_0 \cdot \mathbf{F}_{\chi e} \\ \mathbf{B}_0 \cdot \mathbf{V}_{\parallel, i} + \mathbf{B}_0 \cdot \mathbf{R}_{\chi i} + \mathbf{B}_0 \cdot \mathbf{F}_{\chi i} \end{bmatrix}. \]

The FSA parallel viscous stress and heat stress forces are
The collisional friction and heat friction forces are
\[
\begin{bmatrix}
(\mathbf{B}_0 \cdot \nabla \cdot \tau_{ee}) \\
(\mathbf{B}_0 \cdot \nabla \cdot \Theta_{ee})
\end{bmatrix}
= \frac{m_p \mu_e}{\tau_{ee}} \mathbf{M}_e \cdot \begin{bmatrix}
(B_0^2) U_{e,\theta} \\
(B_0^2) Q_{e,\theta}
\end{bmatrix}.
\]  
(18)

The reference electron-electron collision frequency is
\[
\frac{1}{\tau_{ee}} = \frac{4}{3 \sqrt{\pi}} \left( 4 \pi \rho_0^2 m_e \nu_e \right)^{1/2} \left[ \left( T_e(\nu_e) \right)^{1/2} \right] \left( B_0^2 \right)^{1/2}.
\]
(20)

Finally, the FSA parallel “external” forces and heat forces on the electrons are
\[
\begin{bmatrix}
(\mathbf{B}_0 \cdot \mathbf{F}_{e,\parallel}) \\
\mathbf{F}_{e,\parallel}
\end{bmatrix}
= \begin{bmatrix}
-n_{e,\theta}(\mathbf{B}_0 \cdot \mathbf{F}_{\parallel}) + (\mathbf{B}_0 \cdot \mathbf{S}_{\parallel}^0) \\
\mathbf{F}_{e,\parallel}
\end{bmatrix} \left( \frac{m_e}{\tau_{e,\parallel} B_0^2} \right) \begin{bmatrix}
\mathbf{S}_{\parallel}^0
\end{bmatrix}.
\]
(23)

The sources of electron momentum and heat flow are composed of both direct and fluctuation-induced sources: \( \langle \mathbf{B}_0 \cdot \mathbf{S}_{\parallel}^0 \rangle = \langle \mathbf{B}_0 \cdot \mathbf{S}_{\parallel}^0 \rangle + \langle \mathbf{B}_0 \cdot \mathbf{S}_{\parallel} \rangle \) and \( \langle \mathbf{B}_0 \cdot \mathbf{S}_{\parallel}^0 \rangle = \langle \mathbf{B}_0 \cdot \mathbf{S}_{\parallel} \rangle \) + \( \langle \mathbf{B}_0 \cdot \mathbf{F}_{\parallel} \rangle \). The net electron nonlinear source is \( \langle \mathbf{B}_0 \cdot \mathbf{S}_{\parallel}^0 \rangle = \langle \mathbf{B}_0 \cdot \mathbf{S}_{\parallel} \rangle - m_e \langle \nabla_e \mathbf{V}_{\parallel} \rangle \) and the net inductive source of electron heat flow. The fluctuation-induced electron momentum source is \( \langle \mathbf{B}_0 \cdot \mathbf{S}_{\parallel} \rangle = -\mathbf{B}_0 \cdot (m_e \nu_e \nabla_e \mathbf{V}_{\parallel} + \nabla_e \cdot \mathbf{S}_{\parallel}^0) \) and the analogous heat flow source is
\[
\langle \mathbf{B}_0 \cdot \mathbf{F}_{\parallel} \rangle = -\mathbf{B}_0 \cdot \nabla_e (\nabla_e \mathbf{V}_{\parallel} + \mathbf{S}_{\parallel}^0 + \mathbf{F}_{\parallel} - \nabla_e \cdot \mathbf{I})
\]
which involves many apparently small, rarely calculated or measured quantities.
\begin{align}
\langle \mathbf{B}_0 \cdot \mathbf{J}_{\theta 0} \rangle &= b_{00} - \int \frac{dP_0}{d\psi_p} + n_e e \langle \mathbf{B}_0 \cdot \mathbf{U}_{i\theta} \rangle + b_{01} \left( \frac{dT_{\theta 0}}{d\psi_p} \right), \\
\langle \mathbf{B}_0 \cdot \mathbf{J}_{\phi} \rangle &= \frac{\langle \mathbf{B}_0 \cdot \mathbf{S} \rangle_{\psi v} + c_{01} (m_j / T_{\theta 0}) \langle \mathbf{B}_0 \cdot \mathbf{S} \rangle_{eq}}{n_e e \eta \psi_p}, \\
\langle \mathbf{B}_0 \cdot \mathbf{J}_{\psi} \rangle &= \frac{\langle \mathbf{B}_0 \cdot \mathbf{F} \rangle_{\psi v} + c_{01} (m_j / T_{\theta 0}) \langle \mathbf{B}_0 \cdot \mathbf{F} \rangle_{eq}}{n_e e \eta \psi_p}.
\end{align}

Here, the coefficients are $b_{00} = \langle \mathbf{N}_i + \mathbf{M}_{j} \rangle_{\psi v} \cdot \langle \mathbf{B}_0 \cdot \mathbf{S} \rangle_{eq}$, $b_{01} = \langle \mathbf{N}_i + \mathbf{M}_{j} \rangle_{\psi v} \cdot \langle \mathbf{B}_0 \cdot \mathbf{S} \rangle_{eq}$, and $c_{01} = \langle \mathbf{N}_i + \mathbf{M}_{j} \rangle_{\psi v} \cdot \langle \mathbf{B}_0 \cdot \mathbf{F} \rangle_{eq}$.

The neoclassical offset poloidal ion flow coefficient is given by

\begin{equation}
\mu_{i0} = \frac{1}{\mu_{i00} + (\mu_{i11} - \mu_{i01} / \mu_{i00}) / \mu_{i11}}. \tag{36}
\end{equation}

For a pure electron-ion plasma (i.e., $Z_e \to 0$) in the asymptotic ion banana collisionality regime ($\nu_i \ll 1$, $\varepsilon \ll 1$), this yields the usual results: $\mu_{i01} / \mu_{i00} = 1.17$ and $k_i = 1.17 / (1 + 0.67 \varepsilon)$. Possible poloidal flows driven by external sources and fluctuations are

\begin{equation}
U_{i\theta} = \frac{c_{de} I_{\theta}}{q_0 \langle B_0 \rangle} \frac{\langle B_0 \cdot \mathbf{S} \rangle_{eq}}{n_e e \eta \psi_p} + U_{i\theta}^{\text{drives}}. \tag{37}
\end{equation}

As explained in the paragraph preceding Eq. (24), the fluctuating flows here are fluid flows that result from a combination of guiding center and diamagnetic flows.

To first order in the gyroradius expansion the FSA equilibrium parallel hot flow equation is

\begin{equation}
0 = - \langle \mathbf{B}_0 \cdot \mathbf{V} \rangle_{\theta 0} + \langle \mathbf{B}_0 \cdot \mathbf{U}_{i\theta} \rangle + (m_j / T_{\theta 0}) \langle \mathbf{B}_0 \cdot \mathbf{S} \rangle_{eq}. \tag{33}
\end{equation}

Here, we have $\langle \mathbf{B}_0 \cdot \mathbf{S} \rangle_{eq} = \langle \mathbf{B}_0 \cdot \mathbf{S} \rangle + (5 / 2) p_0 \mathbf{V}_i \cdot \mathbf{V}_i - \langle \mathbf{B}_0 \cdot \mathbf{V} \rangle_{\theta 0} (\mathbf{q} \cdot \mathbf{V} + (2 / 3) \mathbf{V}_i \cdot \mathbf{q} \mathbf{I})$. Since the heat friction

\begin{align}
\langle \mathbf{B}_0 \cdot \mathbf{R} \rangle = - (m_j / T_{\theta 0}) (2 / 5 n \mathbf{q} \mathbf{T}_0) \langle \mathbf{B}_0 \cdot \mathbf{q} \rangle, \tag{32}
\end{align}

and the ion version of Eq. (18) this equation can be solved to yield

\begin{equation}
Q_{i\theta} = - c_{u} U_{i\theta} - \frac{c_{fT}}{q_0 \langle B_0 \rangle} \frac{dT_{\theta 0}}{d\psi_p} + \frac{\tau_{i0} \langle \mathbf{B}_0 \cdot \mathbf{S} \rangle_{eq}}{m_j (v_{i11} + \mu_{i11})}. \tag{34}
\end{equation}

Here, $c_{u} = \mu_{i01} / (v_{i11} + \mu_{i11})$ and $c_{fT} = v_{i11} / (v_{i11} + \mu_{i11})$. Substituting this result into Eq. (32) yields

\begin{equation}
U_{i\theta} = k \frac{I}{q_0 \langle B_0 \rangle} \frac{dT_{\theta 0}}{d\psi_p} + U_{i\theta}^{\text{drives}}. \tag{35}
\end{equation}

IV. MAGNETIC FLUX TRANSIENT EFFECTS

The poloidal and toroidal magnetic fluxes $\psi_p$ and $\psi_t$ evolve during plasma start-up, addition of CDs, and approach to steady state on magnetic field diffusion time scales. These “slow” $O(t^2)$ effects have been negligible in the preceding $O(t^3)$ analyses, but need to be included in the comprehensive transport equations. Using $\mathbf{B} = \nabla \times \mathbf{A}$ with $\mathbf{A} = \psi_p \mathbf{V} - \psi_t \mathbf{\zeta}$ in Faraday’s law the form $\mathbf{V} \times (\nabla / \partial t - \nabla \psi_p + \mathbf{E}) = 0$ and the FSA of $R^{-2}$ times these equations yield the toroidal and poloidal flux evolution equations,

\begin{align}
\frac{\partial \psi_p}{\partial t} \bigg|_x &= - \tilde{u}_G \frac{\partial \psi_p}{\partial \rho} = \psi_t, \tag{39}
\frac{\partial \psi_t}{\partial t} \bigg|_x &= \langle \mathbf{B}_0 \cdot \mathbf{E} \rangle - \frac{\partial \Psi}{\partial t} - \tilde{u}_G \frac{\partial \psi_p}{\partial \rho}. \tag{40}
\end{align}

Here, $\tilde{u}_G = (u_G \cdot \nabla) \mathbf{V} = \langle \mathbf{B}_0 \cdot \mathbf{E} \rangle (\psi_p / (\psi_t^2))$ is the $\psi_t$ “grid speed” and $2 \pi \partial \Psi / \partial t = \mathbf{V}_\text{loop}(t)$ is the toroidal loop voltage.

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induced by the Ohmic heating solenoid. Using the parallel Ohm’s law in Eq. (26) for \(B_0 \times \vec{E} = \nabla \times \vec{B}_0\) and Eqs. (15) and (16) for the parallel current \(\mu_0 (B_0 \cdot \vec{J})\) yields a diffusion equation for poloidal flux \(\psi_p\) on a toroidal flux surface \(\psi_t\),

\[
\dot{\psi}_t = \dfrac{\partial \psi_p}{\partial t} \bigg|_{\psi_t} = D_n \Delta^+ \psi_p - S_p, \quad (41)
\]

\[
D_n = \dfrac{\eta_p}{\mu_0}, \quad S_p = \dfrac{\partial \psi_t}{\partial t} + \left( \frac{\eta_p}{R^2} \right) (B_0 \cdot \vec{J}_{\text{driven}}). \quad (42)
\]

Tokamak plasma properties are determined in terms of the poloidal magnetic flux \(\psi_p\): (1) the Grad–Shafranov equation determines \(\psi_p(x)\) given \(P(\psi_p)\) and \(I(\psi_p)\); (2) classical and neoclassical transport are determined across poloidal flux surfaces \(\psi_p\); and (3) the drift-kinetic and gyrokinetic equations use poloidal flux variables and \(f_0 = f_{\text{Max}}(\psi_p)\) so the canonical toroidal angular momentum emerges as a natural constant of motion. Thus, we need to transform the fluid moment equations from determining the species density, momentum, and energy at a laboratory position \(x\) to determining them on a poloidal flux surface \(\psi_t = \text{const.}\) i.e., \(\partial n_i / \partial \psi_t \Rightarrow \partial n_i / \partial \psi_t\), etc. However, for low collisionality tokamak plasmas this transformation should first be made in the drift-kinetic (or gyrokinetic) equation. This transformation adds a paleo-classical diffusion-type operator \(D(f) \sim D_n f / a^2 \sim \nu \beta f\) to the right side of Eq. (1).

The effects of the paleoclassical transport operator \(D\) will be illustrated through their influence on the perturbed, \(\xi\)-averaged and FSA density equation (2). First, we note that

\[
\langle D(n_0) \rangle = - \rho_p \dfrac{\partial n_0}{\partial \rho} + \left( \nabla \cdot n_0 \vec{u}_G \right) + \left( \frac{\partial^2}{V' \partial \rho^2} (V' \nabla \cdot \vec{D} n_0) \right). \quad (43)
\]

Here, \(\rho_p \equiv \psi_p / \psi_p'\) with \(\psi_p' = d \psi_p / d \rho\) represents \(\psi_p\) surface motion relative to the \(\psi\)-based radial coordinate \(\rho\). \(\nabla_\rho = D_{\rho} / a^2\), \(1 / a^2 = (\nabla_\rho \cdot R^2) / (R^2) = 1 / a^2\) and \(\nabla \cdot \vec{u}_G\) = \((V' / V) (\nabla' / \partial \rho') \rho_0\). Including these transformation effects, the FSA density equation becomes

\[
\left. \dfrac{\partial}{V' \partial \rho} \right|_{\psi_p} \left( V' n_0 \right) + \rho_p \dfrac{\partial n_0}{\partial \rho} + \left( \frac{\partial}{V' \partial \rho} (V' \Gamma) \right) = \langle \vec{S}_\rho \rangle. \quad (44)
\]

The quantity \(V' n_0\) is the number of particles between the \(\rho\) and \(\rho + d \rho\) surfaces; it is an adiabatic plasma property. The radial coordinate \(\rho = \psi_\rho / n B_\rho\) in transport codes is based on the relatively immobile2,3,13 toroidal magnetic flux \(\psi_t\); however, the fluid moments \(n, T, V\) are determined on poloidal flux surfaces \(\psi_t\). The \(\rho_p \partial n_0 / \partial \rho\) term takes account of the slow (magnetic field diffusion time scale) motion of the \(\psi_t\) surfaces relative to the \(\psi_\rho\) surfaces. The total \(O(\delta^2)\) particle flux for each species is

\[
\Gamma = \langle \vec{J} \times \nabla \rho \rangle = \Pi^\rho + \Gamma^{nu} + \Gamma^{pc}. \quad (45)
\]

Here, \(\Gamma^{nu} = n_0 (\nabla_2 - \vec{u}_G) \cdot \nabla \rho\) is the intrinsically ambipolar particle flux driven by axisymmetric collisional processes, \(\Gamma^{pc} = \langle \vec{n}_1 \nabla_1 \cdot \nabla \rho \rangle\) is the possibly nonambipolar fluctuation-induced particle flux, and \(\Gamma^{\rho} = (\partial / \partial \rho) (V' \nabla \rho)\) is the ambipolar paleoclassical particle flux that results from the coordinate transformation.

V. RADIAL PARTICLE FLUXES, PLASMA TOROIDAL ROTATION, AND ELECTRIC FIELD

In Secs. II and III the \(O(\delta^3)\) radial and \(O(\delta^3)\) parallel components of the species force balance equation (3) were analyzed. The \(O(\delta^3)\) particle fluxes will be determined from the toroidal angular component of the force balance equations. A vector identity for determining the second order radial fluxes is

\[
\mathbf{e}_t \cdot n \mathbf{V} \times \mathbf{B}_0 = - n \mathbf{V} \cdot \mathbf{e}_t \times \mathbf{B}_0 = (n \mathbf{V} \cdot \nabla \rho) \psi_p' \quad (46)
\]

Thus, the \(\mathbf{e}_t\) component of the force balance shows the particle flux is induced by \(J\) toroidal torques \(T_{ij} = \mathbf{e}_t \cdot \mathbf{F}_j\) on the plasma species by the various fluid forces \(\mathbf{F}_j\).

\[
0 = \mathbf{e}_t \cdot \left( n \mathbf{q} \mathbf{V} \times \mathbf{B}_0 + \sum_j \mathbf{F}_j \right) \Rightarrow q \psi_p' \Gamma = - \sum_j T_{ij}. \quad (47)
\]

Taking the toroidal angular (\(\mathbf{e}_t \cdot \mathbf{n}\)) component of the species force balance, transforming from \(x\) to \(\psi_t\), and averaging over \(\xi\) and a flux surface yields a particle flux with 15 components.7 There are six collision-induced intrinsically ambipolar fluxes due to toroidal torques from cross friction (classical), parallel friction (Pfirsch–Schlüter), parallel viscosity (banana-plateau), noninductive current drives, dynamos, and the \(\mathbf{E} \times \mathbf{B}_0\) pinch, plus the paleoclassical one due to the coordinate transformation. In addition, there are eight possibly nonambipolar fluxes due to toroidal torques exerted on plasma species by polarization flows induced by \(\delta \Omega_n / \partial \rho \neq 0\), neoclassical toroidal viscosity (NTV) from small 3D NA magnetic fields, perpendicular viscosities (classical, neoclassical, and paleoclassical), Reynolds and Maxwell stresses induced by fluctuations, \(\mathbf{J} \times \mathbf{B}\) forces in the vicinity of low order rational surfaces induced by resonant field errors, poloidal flux transients, and external momentum sources.

For quasineutrality the transport time scale charge continuity equation requires \((\mathbf{J} \cdot \nabla \rho) = 0\). Setting the \((\mathbf{J} \cdot \nabla \rho)\) induced by the nonambipolar fluxes to zero yields a comprehensive toroidal toroidal balance equation for the toroidal angular momentum density \(L_t = m n_0 (R^2 \Omega_n)\) (neglecting small electron terms),

\[
\left. \dfrac{\partial}{V' \partial \rho} \right|_{\psi_p} \left( V' L_t \right) = - \langle \mathbf{e}_t \cdot \nabla \cdot \mathbf{J} \rangle - \langle \mathbf{e}_t \cdot \nabla \cdot \mathbf{B} \rangle
\]

\[
- \rho_p \dfrac{\partial L_t}{\partial \rho} - \left( \mathbf{e}_t \cdot \sum_i \mathbf{S}_i \right). \quad (48)
\]

Here, \(V' L_t\) is the plasma toroidal angular momentum between the \(\rho\) and \(\rho + d \rho\) flux surfaces, which is an adiabatic quantity. The terms on the right represent NTV effects, collision-induced perpendicular viscosity, Reynolds and Maxwell stresses induced by fluctuations, poloidal flux tran-
sients, and externally supplied momentum sources.

Once this equation is solved for $L$, the toroidal rotation is given by $\langle \Omega_\theta \rangle = L/(m_n\bar{n}_b(R^2))$. Hence from Eq. (38) the radial electric field $E_r = -[\nabla \rho (\partial \Phi_0/\partial \rho)]$ is

$$E_r = \nabla \rho \left[ \left( \langle \Omega_\theta \rangle - \langle R^2 \Omega_{\theta r} \rangle \right)_e \rho_\theta + \frac{1}{n_\theta \bar{q}_i} \right] \partial \rho_0 / \partial \rho. \quad (49)$$

This $E_r$ (or $\Omega_\theta$) causes the electron and ion nonambipolar radial particle fluxes to become equal (i.e., ambipolar): $\Gamma_e^{\text{amb}}(E_r) = Z_0 \Gamma_i^{\text{amb}}(E_r)$—to satisfy $\langle J \cdot \nabla \rho \rangle = 0$, which was used to obtain the $\Omega_\theta$ (or $E_r$) equation. Hence, the net ambipolar particle flux is the sum of the intrinsically ambipolar fluxes $\Gamma^a_e + \Gamma^a_i$ and the potentially nonambipolar fluxes evaluated at the electric field $E_r$, $\Gamma^{\text{non}}(E_r)$. It is easiest to evaluate for electrons since the dominant nearly canceling terms in the $\langle J \cdot \nabla \rho \rangle$ that is set to zero to obtain Eq. (48) result from nonambipolar ion particle fluxes (this is the so-called “ion root”$^{16}$).

$$\Gamma^\text{net}_e = \Gamma^a_e + \Gamma^a_i + \Gamma^{\text{non}}(E_r) = \Gamma^\text{net}_i. \quad (50)$$

This is the net ambipolar particle flux $\Gamma$ in the density continuity equation on the $\rho$ flux surface given in Eq. (44). See Eqs. (87)–(93) in Ref. 7 for the seven intrinsically ambipolar particle fluxes and Eqs. (106)–(113) there for the eight potentially nonambipolar particle fluxes.

**VI. ENERGY TRANSPORT EQUATIONS**

The energy transport equations on the transport time scale ($\partial / \partial t ~ \sim \bar{\nabla}$) are obtained by expanding all quantities in Eq. (4) using perturbation expansions like that in Eq. (6). Then, we take the toroidal angle ($\xi$) average of the equations. Finally, we flux surface average the resultants equations and transform them from laboratory ($\mathbf{x}$) to poloidal flux ($\phi_p$) coordinates to obtain for each species

$$\frac{3}{2} \rho \frac{\partial}{\partial \tau} \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \rho V^5 \frac{1}{3} \right] + \frac{3}{2} \phi_p \frac{\partial}{\partial \tau} \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho V^5 Y \right] = Q_{\text{net}}. \quad (51)$$

Here, $\ln(p_i\rho V^5)$ is the collisional entropy between the $\rho$ and $\rho + d\rho$ flux surfaces; without dissipation or energy sources it is conserved. The $\rho \phi_p$ term accounts for motion of $\phi_p$ surfaces relative to the $\phi_i$ surfaces.

The total FSA “radial” heat fluxes are

$$Y = \left[ \frac{q^2}{2} \left( \frac{\partial}{\partial \rho} \rho \nabla \cdot \overline{\mathbf{V}_2} \right) + \frac{1}{V'} \frac{\partial}{\partial \rho} \left( V' \left( \overline{\mathbf{e}_2} \cdot \left( \overline{\mathbf{V}_2} \mathbf{q}_2 + \mathbf{q}_2 \overline{\mathbf{V}_2} + \frac{1}{2} \mathbf{V}_2^2 \mathbf{q}_2 \mathbf{I} \right) \cdot \nabla \rho \right) \right) \right] \nabla \rho. \quad (52)$$

which is comprised of conductive plus convective heat fluxes. The particle flux $\Gamma$ in the convective heat flux $(5/2)T_0 \Gamma$ is the net ambipolar one specified in Eq. (50). The second order conductive heat flux $\mathbf{q}_2$ is obtained similarly to how the radial particle fluxes were obtained via Eq. (46) from the toroidal angular component of the perturbed, $\xi$-averaged, FSA and transformed heat flow equation (5).

$$\langle \mathbf{q}_2 \cdot \nabla \rho \rangle = Y_{\text{cl}} + Y_{\text{PS}} + Y_{\text{bp}} + Y_{\text{pc}} + Y_{\text{sq}}. \quad (53)$$

The lowest order classical (cross heat friction), Pfirsch–Schlüter (parallel heat friction), banana-plateau (parallel heat viscosity) and paleoclassical (transform to $\phi_p$) collision-induced heat fluxes, and fluctuation- and source-induced conductive radial heat fluxes obtained from the $\mathbf{e}_c = R^2 \nabla \xi$ component of Eq. (5) for each species are

$$Y_{\text{cl}} = T_0 \left( \mathbf{B}_0 \times \frac{\nabla \rho}{q_r \mathbf{B}_0} \cdot \overline{\nabla} S_q \right), \quad (54)$$

$$Y_{\text{PS}} = \frac{IT_{\text{ps}}}{q_r \phi_p} \left( \frac{1}{B_0} - \frac{1}{\langle B_0^2 \rangle} \right) \mathbf{B}_0 \cdot \overline{\mathbf{B}}_S_q, \quad (55)$$

$$Y_{\text{bp}} = \frac{IT_{\text{bp}}}{q_r \phi_p} \mathbf{B}_0 \cdot \overline{\mathbf{B}}_S_q, \quad (56)$$

$$Y_{\text{pc}} = \frac{1}{V'} \frac{\partial}{\partial \rho} \left[ \overline{\mathbf{V}}^D \left( \frac{3}{2} \rho \rho_0 T_0 \right) \right], \quad (57)$$

Here, the perpendicular viscosity effects$^7$ have been neglected since they give $\mathcal{O}(\bar{\nabla})^2$ smaller contributions to the heat fluxes. The rather complicated fluctuation-driven contributions in $Y_\Omega$ given by Eq. (58) are usually much smaller than the typically dominant $(5/2)\rho \langle \overline{\mathbf{V}}_1 \cdot \nabla \rho \rangle$ fluctuation-driven contribution to Eq. (52)—because they mostly involve the smaller flow fluctuation magnitudes in the toroidal direction compared to the typically larger ones in the radial direction. As explained in the paragraph preceding Eq. (24),

$$Y_{\text{sq}} = -\frac{m_i}{q_r \phi_p} \langle \mathbf{e}_c \cdot \overline{\mathbf{q}_2} \rangle. \quad (59)$$
the fluctuating flows and heat flows here are fluid flows that result from a combination of guiding center and diamagnetic flows. However, as with the fluctuation-induced particle flux (see the discussion in Appendix A of Ref. 7), the fluctuating diamagnetic flow does not contribute to the fluctuation-induced heat transport; thus the lowest order fluctuation-induced contribution to $Y$ is just due to guiding center flow, i.e., $(5/2)n_0(T_i \nabla_i \rho_p) = (5/2)n_0(T_i \nabla_i \nabla_p)$. Finally, the net rate of energy input to the species $Q_{\text{net}}$ in Eq. (51) is given in general by

$$Q_{\text{net}} = \langle \dot{Q} \rangle + (\langle \nabla_2 - u_e \cdot \nabla \rangle \cdot p_0) + (\langle \nabla_1 \cdot \nabla \rangle \cdot p_0) + (\langle \nabla_1 \cdot \nabla \rangle \cdot \dot{p}_1) - \langle \nabla \cdot \nabla \nabla_1 \rangle + \langle S_{\text{FSA}} \rangle. \quad (60)$$

Successive terms on the right will be specified in greater detail in the following paragraphs.

The total collisional energy exchange between species is

$$Q = \int d^3v (m_v \nu \nu^2/2) C(f) = \int d^3v (m_v \nu^2/2 - m_v \nu \cdot \nu) C(f),$$

with $\nu$ the electron energy equation because of collisional energy conservation. As discussed in the next paragraph, the second term represents primarily Joule heating. The final term in Eq. (61) represents fluctuation-induced parallel and perpendicular Joule heating; it has been found to be negligible for some typical tokamak plasma conditions.

The net collisional Joule heating rate can be obtained by neglecting the last term in Eq. (61) and summing over species: $\langle \dot{Q}_J \rangle = \sum_i \langle \dot{Q}_J \rangle = -\sum_i (\nabla_3 \cdot \nabla_3 \nabla_i)$. The first order flows and currents can be represented in terms of their parallel and cross components by

$$V_{sl} = U_{sl}(\psi_p) (B_0 + \Omega_{\psi_{sl}}) \psi_e, \quad (62)$$

$$\Omega_{\psi_{sl}} = -\left( \frac{d\phi_p}{d\psi_p} + \frac{1}{n_{s0} q_s} \frac{dp_{s0}}{d\psi_p} \right), \quad (63)$$

$$J = \sum_i n_i q_i \nabla_3 = K_i (\psi_p) B_0 - \frac{dP_0(\psi_p)}{d\psi_p} e_t. \quad (64)$$

As noted before Eq. (14), $\langle B_0 \rangle$ is $K_i (\psi_p) B_0 + I_d p_0 / d\psi_p$. Also, we find from Ampere's law $\mu_0 J = \nabla \times B_0 = \nabla I \times \nabla \xi + \delta \psi_p \nabla \nabla \xi$, that $K_i = J \nabla \psi_p / B_0 \nabla \psi_p = -\mu_0 / (1 / \mu_0) d I / d \psi_p$. Thus, the toroidal and poloidal components of the current are $(B_0 \hat{\nabla}) = \Omega / R^2$,

$$\frac{\langle J \cdot \nabla \xi \rangle}{B_0} = \frac{\langle J \cdot \nabla \xi \rangle}{B_0} - \frac{dP_0 / d\psi_p}{B_0} \left( \frac{1}{1 - \frac{R^2}{\Omega^2}} \right), \quad (65)$$

$$J \cdot \psi_p = \frac{K_i (\psi_p) B_0 - 1 \alpha L}{\mu_0 d \psi_p}. \quad (66)$$

Note that the toroidal current $\nabla \cdot \nabla \xi$ has both a FSA component and a (small) Pfirsch–Schlüter type component. The poloidal current $\nabla \cdot \nabla \xi$ is typically smaller than the toroidal current by a factor of $B_0 / B_0 \sim e / q$.

Using these flow and current representations, the first row of Eq. (17) for the FSA parallel electron friction force $(\langle B_0 \rangle \nabla \xi) = (\langle B_0 \rangle \nabla \xi) / (e \xi \cdot \xi) + (u_{G} \cdot \nabla \xi)$, and from collisional momentum conservation $\rho_{visc} = -R_{visc}$, we find for the Joule heating

$$\langle \dot{Q}_J \rangle = -\sum_i (U_{sl} B_0 \cdot \nabla_i) + \Omega_{\psi_{sl}} \langle e \xi \cdot R_i \rangle$$

$$= K_r \left[ \frac{\langle B_0 \cdot R_i \rangle}{\n_{s0 e}} - \frac{\Gamma_r}{\n_{s0 e}} \frac{\langle B_0 \cdot \xi \rangle}{\n_{s0 e}} \right] - \frac{\Gamma_r}{\n_{s0 e}} \frac{\langle B_0 \cdot \xi \rangle}{\n_{s0 e}} \frac{\langle B_0 \cdot \psi_p \rangle}{\n_{s0 e}}, \quad (67)$$

The $P_{d \pi V}$ work by the collision-induce-induced classical, Pfirsch–Schlüter and banana-plateau particle flux $\Gamma_r$ is usually negligible. For Ohmically heated tokamak plasmas where the Joule heating from the average toroidal current induced by the toroidal inductive electric field usually dominates over that induced by the poloidal current (for $B_0 / B_0 \sim e / q < 1$), we obtain simply $\langle \dot{Q}_J \rangle = (\langle B_0 \rangle \nabla \xi / \langle B_0 \rangle / \langle B_0 \rangle) / \langle B_0 \rangle$. The FSA inductive parallel electric field $(\langle B_0 \rangle \nabla \xi / \langle B_0 \rangle)$ is provided by the extended parallel classical Ohm's law given in Eq. (26).

Implicitly, the Joule heating result in Eq. (67) is obtained in the ion rest frame. Thus, for ions we have

$$\langle \dot{Q}_J \rangle = \langle \dot{Q}_J \rangle - \langle \nabla_3 \cdot \nabla_3 \nabla \rangle. \quad (68)$$

The corresponding collisional energy exchange rate for electrons is

$$\langle \dot{Q}_J \rangle = \langle \dot{Q}_J \rangle - \langle \nabla_3 \cdot \nabla_3 \nabla \rangle. \quad (69)$$

The second $Q_{\text{net}}$ term in Eq. (60) represents the $pdV$ work done by an outflowing species of particles. For either species it is

$$\langle (\nabla_3 - u_{G} \cdot \nabla \pi) p_{s0} \rangle = (\Gamma_{s0}/n_{s0})(dp_{s0}/d\rho). \quad (70)$$

This contribution cancels part of the small $P_{d \pi V}$ work term in Eq. (67). The third $Q_{\text{net}}$ term in Eq. (60) represents $pdV$ work by fluctuations as they induce radial particle transport. Using the vector identity $\nabla \cdot \rho dV = \rho dV - \nabla \cdot \rho \nabla - \rho dV$, it can be seen that when fluctuating flows are incompressible the remaining $(\nabla \cdot \rho dV) = (1 / I_{\pi}) (\delta \rho / \delta \rho)(V \cdot (\rho dV / \rho dV))$ term just changes the $5/2$ factor in the convective heat flow in Eq. (52) to $3/2$, as is well known. The fourth and fifth $Q_{\text{net}}$ terms in Eq. (60) represent viscous heating. Using the vector identity that $\nabla \cdot \rho dV = \nabla \cdot (\nabla \cdot \rho dV - \nabla \cdot \rho dV)$, the fluid velocity representation in Eq. (62) and the fact that $(e \xi \cdot \nabla \xi) = 0$, these viscous heating terms reduce for each species to

$$Q_{\text{visc}} = \langle \nabla_1 \cdot \nabla_1 \nabla_i \rangle - \langle \nabla_3 \cdot \nabla_3 \nabla i \rangle = U_{sl} \langle B_0 \rangle \nabla \cdot \nabla \xi, \quad (71)$$

which is $\sim \mu_0 U_{sl} \langle B_0 \rangle$. However, it just cancels part of the typically small poloidal current contribution to Eq. (67).
For electrons there is one additional term to be added to the left side of Eq. (51). It is the contribution due to helically resonant paleoclassical transport processes,\textsuperscript{18}

\[
\langle \nabla \cdot \mathbf{q}_e^{\text{pe}} \rangle = -\frac{M}{V'} \frac{\partial^2}{\partial \rho^2} \left( V' D \frac{3}{2} \frac{\partial \rho_{e0}}{\partial \rho} \right) + \frac{3}{2} \rho_{e0} \frac{\partial \rho_{e0}}{\partial \rho}.
\]  

(72)

Here, \( M \sim 10 \) is a factor reflecting the distance, relative to the poloidal periodicity length, over which the electron temperature is equilibrated as it diffuses radially with the diffusing poloidal magnetic flux.\textsuperscript{18} Also, \( \rho_{e0} = -q(\partial \psi_p / \partial \rho)/q' \psi_p' \) with \( q' = \partial q / \partial \rho \) reflects the \( \rho \) motion required to stay on the same \( q = (\partial \psi / \partial \rho)/(\partial \psi_s / \partial \rho) \) surface in transient poloidal flux situations where \( \psi_s(\rho, t) = \psi_p(\rho, 0) + \psi_s(\rho) t \). There is no paleoclassical \( M \) contribution to the ion energy balance because\textsuperscript{18} the ion toroidal precessional drift frequency exceeds the ion collision frequency in most tokamak plasmas.

VII. SUMMARY

Comprehensive plasma flow and transport equations for describing tokamak plasmas have been derived with a kinetic-based approach using a small gyroradius expansion. Relevant fluid moment equations that include neoclassical-based kinetic closures and nonaxisymmetric perturbations (externally imposed or from plasma fluctuations) have been averaged over the toroidal angle \( \zeta \) and then FSA. The zeroth order Alfvén time scale radial force balance provides a constraint relation for the toroidal plasma rotation \( \Omega \) in Eqs. (8) and (38). The first order parallel force balances yield the neoclassical-type parallel Ohm’s law in Eq. (26) and poloidal ion flow in Eq. (35) for times longer than collision times. The radial fluxes of particles \( \Gamma \) in Eq. (45) and heat \( Y \) in Eqs. (52)–(59) are obtained from toroidal angular components of the momentum and heat flow equations, respectively. The resultant tokamak plasma model includes transport time scale evolution equations for the toroidal magnetic flux \( \psi_t \) in Eq. (39), poloidal magnetic flux \( \psi_p \) in Eq. (41), and toroidal angular momentum density \( L_t = m_p n_0 (R^2 \Omega) \) in Eq. (48), in addition to expanded versions of the FSA equations for the density \( n \) in Eq. (44) and species pressure \( p = nT \) in Eq. (51). The FSA plasma toroidal rotation frequency \( (\Omega_t) = L_t / (m_p n_0 R^2) \) determines the radial electric field \( E_r \) in Eq. (49) that is required to obtain the net ambipolar radial particle flux \( \Gamma_{\text{net}} \) in Eq. (50).

Key attributes of and consequences from this new approach for tokamak plasma transport equations are the following. (1) The derivation of the radial particle flux and toroidal flow are naturally joined. (2) The radial electric field is determined self-consistently and enforces ambipolar radial particle transport. (3) The “mean-field” effects of microturbulence-induced fluctuations on all plasma transport properties are included self-consistently—parallel Ohm’s law [Eq. (31)], poloidal ion flow [Eq. (37)], particle fluxes [Eq. (45)], toroidal rotation [Eq. (48)], heat fluxes [Eqs. (52) and (58)], and heating [Eqs. (60) and (61)]. (4) Source effects (e.g., energetic neutral beam momentum input and noninductive current drives) are also included self-consistently. (5) Paleoclassical \( n \), \( \Omega_t \), and \( T \) diffusion and pinch effects plus the electron heat transport in Eq. (72) are included naturally. Finally, (6) the radial motion of \( n \), \( \Omega_t \), and \( T \) induced by \( \psi_p \neq 0 \) poloidal flux transients are included for the first time. These plasma transport equations follow naturally from extended two-fluid moment equations; hence they are consistent with extended MHD code frameworks and could provide a basis for the Fusion Simulation Program.

ACKNOWLEDGMENTS

The authors are grateful to G. Bateman who stimulated us to develop the “simplified” Joule heating formula in Eq. (67).

This research was supported by U.S. Department of Energy Grant No. DE-FG02-86ER53218.

10See supplementary material at http://dx.doi.org/10.1063/1.3335486 for multicollionality viscosity coefficient matrices (also available as UW-CPTC 09-6R via http://www.cptc.wisc.edu).