

ECE533 Digital Image Processing Course Project:

Image Restoration using Iterative Wiener Filter

Jing Liu, Yan Wu

December 13, 2003

1 Introduction

Wiener filter, as the minimal mean square error estimator, is often used to restore degraded images. However it requires a priori knowledge of the power spectral density of original images, which is often unavailable in practice. Thus the estimation of power spectral density of the original image from a single copy of the degraded image is a crucial issue that influences the true success of the restoration of the degraded image.

In this project, we implemented an iterative Wiener filter in frequency domain to iteratively estimate the power spectral density of the original image and restore the degraded image. In each iteration, the power spectral density of the original image is estimated using periodogram method. Then the restored image, which has passed through the corresponding Wiener filter, is feed back as an updated estimate of the original image, and leads to a new iteration, until the estimate of the power spectral density converges, and the mean square error reaches its minimal.

We did experiments with a linear degradation models – a low-pass blurring filter. Noise was chosen to be additive white Gaussian noise. Test cases were run under various signal-to-noise-ratios (SNR).

Test results were shown and analyzed. The figures of the mean square error versus the number of iterations in all cases clearly demonstrated the convergence of power spectral density and the decrease of the mean square error. In particular, the results using basic iterative method and additive iterative method were compared. Using additive iterative method, the MSE has an obvious trend to converge to some "optimal" value, where the estimated power spectral density is supposed to converge to its true value. For different SNR applied, the similar results were obtained. In addition, the smaller the SNR is, the slower MSE converges. However using basic iterative method, only when the noise is small, the estimated power spectral density in basic iterative method approaches that in additive iterative method. When the influence of

noise gets larger, the MSE in basic iterative method even exceeds that in the original wiener filter.

Finally, a Graphic User Interface (GUI) was implemented to facilitate the demonstration of the effect of the iterative Wiener image restoration.

This report is organized as below. First, the key idea of iterative Wiener filter will be introduced. Then its implementation will be described in detail. In the next section, experiment results will be shown with corresponding analysis. Then, the problems we met and solved will be briefly mentioned. Finally, a summary will be given and future research directions explored.

2 Motivation and Rationale

First let us briefly review the theory of Wiener filter. The degradation model is given by

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \mathbf{n} \quad (1)$$

where \mathbf{H} is a linear shift degradation model, \mathbf{n} is a white Gaussian noise, \mathbf{f} is the original image, and \mathbf{g} is the degraded image. The original image is assumed as $m \times m$, then \mathbf{H} is a matrix of $m^2 \times m^2$, while \mathbf{g} , \mathbf{f} and \mathbf{n} are vectors of size $m^2 \times 1$. In this report, bold face symbols are used to represent matrices (upper case) and vectors (low case).

Wiener filter is optimal in the sense of mean square error. By applying the orthogonal principle, we can derive spatial domain Wiener filter and the restored image as follows [1]:

$$\mathbf{B} = \mathbf{R}_f \mathbf{H}^H [\mathbf{H} \mathbf{R}_f \mathbf{H}^H + \mathbf{R}_n]^{-1} \quad (2)$$

$$\hat{\mathbf{f}} = \mathbf{B}\mathbf{g} \quad (3)$$

where \mathbf{B} is wiener filter in spatial domain, \mathbf{R}_f and \mathbf{R}_n are autocorrelation matrices of \mathbf{f} and \mathbf{n} respectively, $(\cdot)^H$ denotes the conjugate transpose and $\hat{\mathbf{f}}$ denotes the restored image. It should be noted that the above result was derived under three basic *assumptions*:

1. The original image and noise are statistically independent, which makes sense especially when noise is additive white Gaussian noise.

2. The power spectral density (or autocorrelation) of the original image and noise are known. The power spectral density of the original image is a priori knowledge that could be attained from previous measurements of similar images. However in practice, sometimes only a single copy of degraded image is available. The estimate of the power spectral density of the original image from the single copy of degraded image is not only necessary, but also crucial to the effect of restoration. That is also the *motivation* for the iterative Wiener filter method.

3. Both original image and the noise are zero mean. If the image is not zero mean, we need to subtract its mean value before it passes through the Wiener filter, and add the mean thereafter.

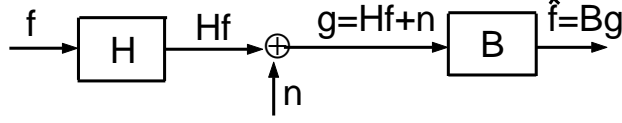


Figure 1: System Model

The system model is shown in Figure 1.

The *rationale* of iterative Wiener filter is to use the restored image (after passing through Wiener filter) as an improved prototype of the original image, estimate its power spectral density and construct new Wiener filter. More specifically, it uses the degraded image as an initial estimate of the original image, and attains a restored image accordingly. This restored image is then used as an updated estimate of the original image and leads to a new restoration. The iterations continue until convergence. Intuitively, every iteration provides a better estimate. And actually it can be approved.

An iterative procedure is listed below:

0. Initialization: $\mathbf{R}_f(0) = \mathbf{R}_g$, where \mathbf{R}_g is the autocorrelation matrix of \mathbf{g}
1. Wiener filter construction: $\mathbf{B}(i+1) = \mathbf{R}_f(i)\mathbf{H}^H [\mathbf{H}\mathbf{R}_f(i)\mathbf{H}^H + \mathbf{R}_n]^{-1}$
2. Restoration: $\hat{\mathbf{f}}(i+1) = \mathbf{B}(i+1)\mathbf{g}$
3. Update: $\mathbf{R}_f(i+1) = E\{\hat{\mathbf{f}}(i+1)\hat{\mathbf{f}}(i+1)\}$

Step 1 to 3 are repeated until the result converges.

The above iterative procedure is provided in spatial domain, however, the large computation makes it impractical. In the following, we will deduce the iterative Wiener filter in discrete Fourier domain.

Basic iterative algorithm

Applying the update and iteration shown above, we get

$$\begin{aligned}
 \mathbf{R}_f(i+1) &= E\{\hat{\mathbf{f}}(i+1)\hat{\mathbf{f}}(i+1)\} \\
 &= \mathbf{B}(i+1)\mathbf{R}_g\mathbf{B}^H(i+1) \\
 &= \mathbf{R}_f(i)\mathbf{H}^H [\mathbf{H}\mathbf{R}_f(i)\mathbf{H}^H + \mathbf{R}_n]^{-1}\mathbf{R}_g[\mathbf{H}\mathbf{R}_f(i)\mathbf{H}^H + \mathbf{R}_n]^{-1}\mathbf{H}\mathbf{R}_f^H(i) \quad (4)
 \end{aligned}$$

Since the updating autocorrelation \mathbf{R}_f is a *matrix* of size $m^2 \times m^2$, it has a very high complexity of computation to do the updating. To process a 256×256 black-white image, you need to

compute addition, multiplication, conjugate transpose and even inverse of some matrices of size $256^2 \times 256^2$! To avoid that huge computation, the 2D discrete Fourier transform is applied to diagonalize above matrices.

The diagonalization of circulant matrix using 2D discrete Fourier transform is shown in Appendix. So once we can approximate circulant matrices of those block Teoplitz matrices \mathbf{R}_f , \mathbf{R}_g , \mathbf{R}_n and \mathbf{H} , we can reduce the computation of matrices to that of scalars (only the elements in the diagonals of the matrices appear.) Let \mathbf{W} denote the matrix of 2D discrete Fourier transform and we can derive:

$$\begin{aligned} & \mathbf{W}\mathbf{D}_f(i+1)\mathbf{W}^H \\ &= \mathbf{W}\mathbf{D}_f(i)\mathbf{D}_h^H [\mathbf{D}_h\mathbf{D}_f(i)\mathbf{D}_h^H + \mathbf{D}_n]^{-1}\mathbf{D}_g[\mathbf{D}_h\mathbf{D}_f(i)\mathbf{D}_h^H + \mathbf{D}_n]^{-1}\mathbf{D}_h\mathbf{D}_f^H(i)\mathbf{W}^H \end{aligned} \quad (5)$$

where \mathbf{D}_f , \mathbf{D}_g , \mathbf{D}_h and \mathbf{D}_n are the diagonalized matrix of \mathbf{R}_f , \mathbf{R}_g , \mathbf{H} and \mathbf{R}_n by taking 2D discrete Fourier transform, respectively. Thus the diagonal elements of \mathbf{D}_f , denoted as p_f , which are actually the power spectral density of \mathbf{f} , satisfy the following scalar equation

$$p_f(i+1) = \frac{p_g p_f^2(i) |p_h|^2}{[p_f(i) |p_h|^2 + p_n]^2} \quad (6)$$

where p_g , p_h and p_n denote diagonal elements of diagonal matrices \mathbf{D}_g , \mathbf{D}_h and \mathbf{D}_n , respectively.

It can be proved that the estimation of the power spectral density is sure to converge, but not to its true value. So a correction item is added in every iteration. It is the so-called additive iterative method.

Additive iterative algorithm

The questions are whether the estimate of the power spectral density converges, and whether it converges to the true value.

It is easy to prove that $p_f(i)$ is sure to converge to some value \tilde{p}_f , because $p_f(i)$ as a sequence is both monotonic (The derivative of $p_f(i+1)$ with respect to $p_f(i)$ is greater than 0) and bounded. \tilde{p}_f can be derived as follows [1]:

$$\tilde{p}_f = \frac{1}{2} \left[p_f - \frac{p_n}{|p_h|^2} \pm \sqrt{p_f^2 - \frac{3p_n^2}{|p_h|^4} - \frac{2p_n p_f}{|p_h|^2}} \right] \quad (7)$$

The problem is that \tilde{p}_f does not converge to p_f unless p_n is 0. So a correction item is added in every iteration, and an additive iterative method was put forward. p_f^* is used as the update of original image instead of p_f . It can be derived that

$$p_f^*(i+1) = p_f^*(i) + \frac{|p_h|^2 p_f^{*2}(i) [|p_h|^2 p_f^{*2}(i) - p_n]}{[p_f^*(i) |p_h|^2 + p_n]^2} \quad (8)$$

3 Implementation

Periodogram

Given a vector signal $\mathbf{x} = [x(1), x(2), \dots, x(N)]^T$ (such as \mathbf{g} and \mathbf{n} , $N = m^2$), its autocorrelation coefficients can be estimated as:

$$r(k) = \begin{cases} \frac{1}{N} \sum_{i=k}^{N-1} x(i)x(i-k) & k \geq 0 \\ r(-k) & k < 0 \end{cases} \quad (9)$$

and the Toeplitz autocorrelation matrix is

$$\mathbf{R}_x = \begin{bmatrix} r(0) & r(1) & \cdots & r(N-2) & r(N-1) \\ r(1) & r(0) & \ddots & r(N-3) & r(N-2) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ r(N-2) & r(N-3) & \cdots & r(0) & r(1) \\ r(N-1) & r(N-2) & \cdots & r(1) & r(0) \end{bmatrix} \quad (10)$$

We can approximate \mathbf{R}_x as a circulant matrix $\tilde{\mathbf{R}}_x$ (notice that there some changes at the upper triangle of the matrix):

$$\tilde{\mathbf{R}}_x = \begin{bmatrix} r(0) & r(N-1) & \cdots & r(2) & r(1) \\ r(1) & r(0) & \ddots & r(3) & r(2) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ r(N-2) & r(N-3) & \cdots & r(0) & r(N-1) \\ r(N-1) & r(N-2) & \cdots & r(1) & r(0) \end{bmatrix} \quad (11)$$

The result of the Fourier diagonalization of circulant matrices (see Appendix) shows that the vector, \mathbf{p}_x , which contains the diagonal elements of the obtained diagonal matrix, is 1D Fourier transform of autocorrelation coefficients $\mathbf{r} = [r(1), r(2), \dots, r(N)]^T$:

$$\mathbf{p}_x = \mathcal{F}_{1D}\{\mathbf{r}\} = \frac{1}{N} |\mathcal{F}_{1D}\{\mathbf{x}\}|^2 \quad (12)$$

which is the power spectrum density of \mathbf{x} estimated using unwrapped periodogram method.

Degradation Model

As discussed before, to reduce the computation complexity, we need to approximate the degradation model \mathbf{H} as a circulant matrix. In our project we can choose to design the

where all the operations of vectors are *element by element*.

Iterative procedure

Finally, we implemented iterative Wiener filter and image restoration in the discrete Fourier domain in the following steps:

1. Generate degraded image \mathbf{g} with degradation model \mathbf{H} and white Gaussian noise with certain SNR, and regard \mathbf{g} as the initial estimate of the original image \mathbf{f} .

2. Subtract the mean value from \mathbf{g} (forced to be zero-mean), take 1D discrete Fourier transform, and get the power spectrum density estimate of \mathbf{g} , i.e. the initial power spectrum density estimate of the original image: $\mathbf{p}_f(0) = \mathbf{p}_g = \frac{1}{m^2} |\mathcal{F}_{1D}\{\mathbf{g}\}|^2$

3. Estimate the power spectral density of noise: $\mathbf{p}_n = \frac{1}{m^2} |\mathcal{F}_{1D}\{\mathbf{n}\}|^2$

4. Calculate $\mathbf{p}_f(i+1)$ Eq.(16), and add the correction item to calculate $\mathbf{p}_f^*(i+1)$ Eq.(17)

5. Calculate Wiener filter $\mathbf{B}(i+1)$ as a circulant matrix with \mathbf{b} as its first column, where \mathbf{b} is the inverse Fourier transform of vector $\mathbf{p}_b(i+1)$:

$$\mathbf{p}_b(i+1) = \frac{\mathbf{p}_f(i)\mathbf{p}_h^H}{\mathbf{p}_f(i)|\mathbf{p}_h|^2 + \mathbf{p}_n} \quad (18)$$

and get restored image $\hat{\mathbf{f}}(i+1) = \mathbf{B}(i+1)\mathbf{g}$; in the meanwhile, calculate the mean square error $\|\mathbf{f} - \hat{\mathbf{f}}(i+1)\|^2$

6. If the mean square error does not converge (or within the given number of iterations), then take the restored image $\hat{\mathbf{f}}(i+1)$ as the updated estimate of the original image, and begin a new iteration from step 4

Graphic User Interface (GUI)

A Graphic User Interface (GUI) was implemented using Matlab to facilitate the demonstration of the effect of the iterative Wiener image restoration.

There are several options to choose:

- The original image from a set of images;
- The image size to be processed (from 32 x 32 to 128 x 128);
- The size of the blurring filter which is the degradation model (from 3x3 to 21 x 21);
- Signal-to-Noise-Ratio (0 to 100 db);

- Number of iterations.

The corresponding results are shown:

- The curves of mean square error versus the number of iterations;
- The set of images (original image, degraded image and restored images).

4 Experiment Design, Result and Analysis

In our project, both color and black-white images are processed. For color images, the RGB components are treated individually. The image size is chosen to be 128×128 . Degradation model \mathbf{H} is of a blurring filter of size 11×11 ($k=11$, see section 3).

We did experiments using both basic iterative method and additive iterative method, with various SNR (10dB, 20dB, 30dB, or 40dB). The number of iterations was set to be 25.

The results of two groups of experiments are illustrated below, where Figure 2 to Figure 3 are related to image of fruits, and Figure 4 and Figure 5 are related to image of face. In each group of experiments, two categories of results are shown:

- The curves of mean square error (MSE) versus the number of iterations, and
- The set of images (original image, degraded image and restored images)

From Figure 2 and Figure 4, it can be observed that using *additive iterative method*, the MSE decreases as the number of iterations increases, and the MSE has *an obvious trend to converge to some "optimal" value*, where the estimated power spectral density is supposed to converge to its true value as derived in section 2. The excellent denoising and deblurring effects can be seen from those restored images (Figure 3 and Figure 5). For different SNR applied, the similar results were obtained. In addition, the smaller the SNR is, the slower MSE converges.

From Figure 2 and Figure 4, it can be observed that using *basic iterative method*, the MSE also converges finally. However as mentioned earlier, only when the noise is zero, the estimated power spectral density in basic iterative method approaches that in additive iterative method. It can be observed that when the influence of noise is small (when SNR is large, for example 40dB), the MSE in basic iterative method approaches that in additive iterative method. When the influence of noise gets larger (as SNR decreases), the MSE in basic iterative method even exceeds that in the original wiener filter (given in the first iteration). *Thus SNR has an important influence on where the MSE converges.*

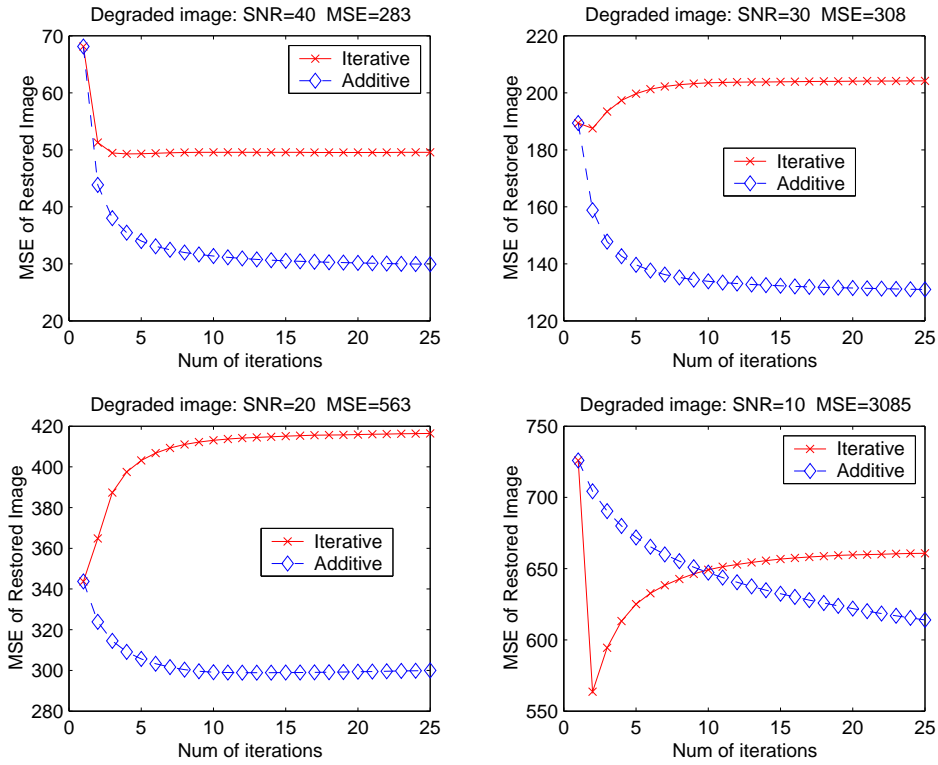


Figure 2: MSE versus number of iterations of fruits image, with SNR=10, 20, 30 and 40 dB.



Figure 3: Fruits Image, SNR=40 dB.

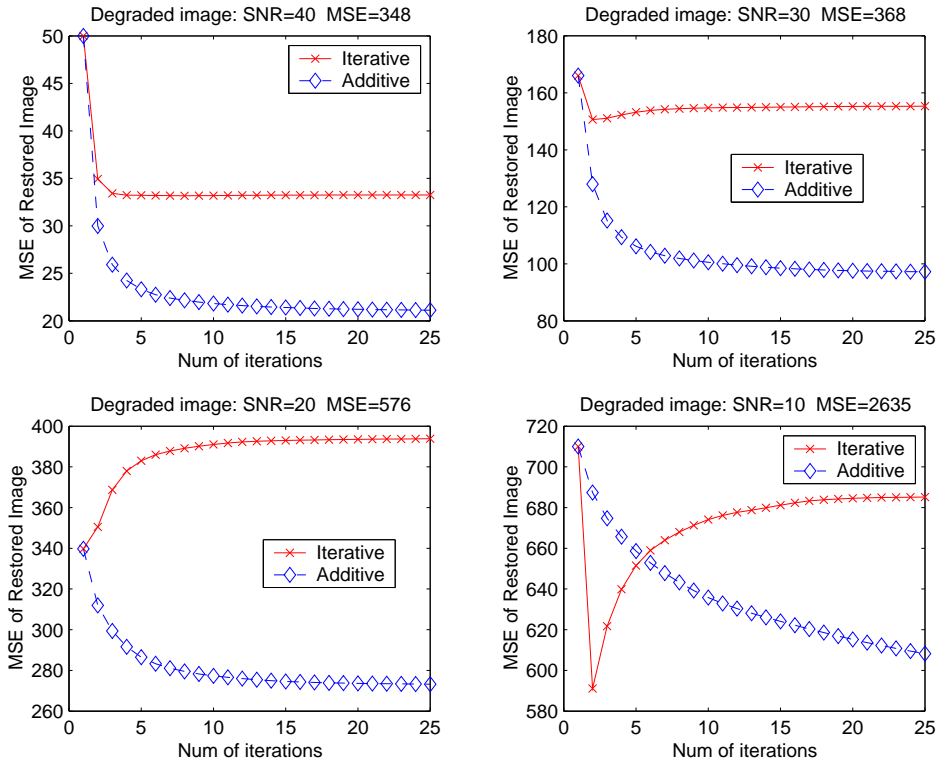


Figure 4: MSE versus number of iterations of face image, with SNR=10, 20, 30 and 40 dB.

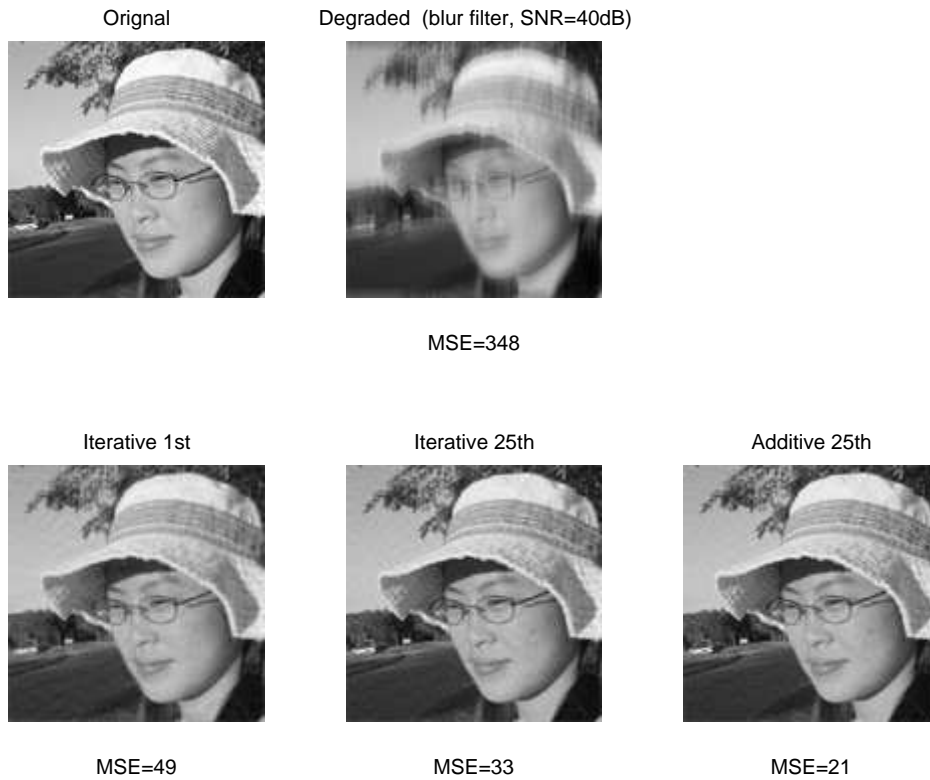


Figure 5: Face Image, SNR=40 dB.

5 Problems we met

One issue is the circulant approximation of the degradation filter h . A particular h was designed (as shown above) to satisfy the requirement.

Another problem is that Wiener filter is assumed to be applied to zero-mean images, which we ignored in the beginning. The result was that iterations didn't take effect (only the first iteration reduced the mean square error). This was because a great number, dc value, was dominant, and

To be more computationally efficient, in the beginning we divided the image into small blocks (32×32) to work on. The result was not satisfactory - there were boundary effect between different blocks. So we processed the whole image in the end.

6 Conclusion

As we have seen, iterative Wiener filter is an effective method to estimate the power spectral density of the original image from a single copy of degraded image. The mean square error decreases with the number of iterations increasing until it converges. This helps to contribute to the true success of the image restoration using Wiener filter.

In this project, we implemented an iterative Wiener filter in discrete Fourier domain to iteratively estimate the power spectral density of the original image and restore the degraded image accordingly. The key idea is to use the restored image as an improved prototype of the original image. Taking discrete Fourier transform diagonalizes the iteration relation from its matrix form to a scalar form, thus greatly reduces the complexity of computation. The power spectral density is estimated using periodogram method. Finally, an additive algorithm was applied to make the estimate converge to its true value.

We did experiments with a degradation model and various signal-to-noise-ratios. We applied both basic iterative filter and additive iterative filter. In all cases, the mean square error converges, and the additive filter has better performance than iterative filter.

It can be observed that using *additive iterative method*, the MSE decreases as the number of iterations increases, and the MSE has an obvious trend to converge to some "optimal" value, where the estimated power spectral density is supposed to converge to its true value as derived in section 2. For different SNR applied, the similar results were obtained. In addition, the smaller the SNR is, the slower MSE converges.

It can be observed that using *basic iterative method*, the MSE also converges. However only when the noise is small, the estimated power spectral density in basic iterative method approaches that in additive iterative method. When the influence of noise gets larger, the MSE in basic iterative method even exceeds that in the original wiener filter. *Thus SNR has*

an important influence on where the MSE converges.

Further researches may include the comparison of the iterative Wiener filter with the constraint least square filter or adaptive filter.

References

- [1] Allen D. Hillary and Ronald T. Chin, "Iterative Wiener filter for image restoration" *IEEE Trans. Signal Processing, Volume: 39 , 1991 Page(s): 1892 -1899.*
- [2] H.C. Andrews and B.R. Hunt, "Digital Image Restoration".
- [3] M. I. Sezan and H. J. Trussell, "Use of a priori knowledge in multispectral image restoration"
- [4] R. Chellappa and R. L.Kashyap, "Digital image restoration using spatial interaction models".

Appendix

appendix-circulant.ps *Fourier Diagonalization of Circulant Matrices*