Lecture 39 Hopfield Network

Outline

- Fundamentals of Hopfield Net
- Analog Implementation
- Associate Retrieval
- Solving Optimization Problem

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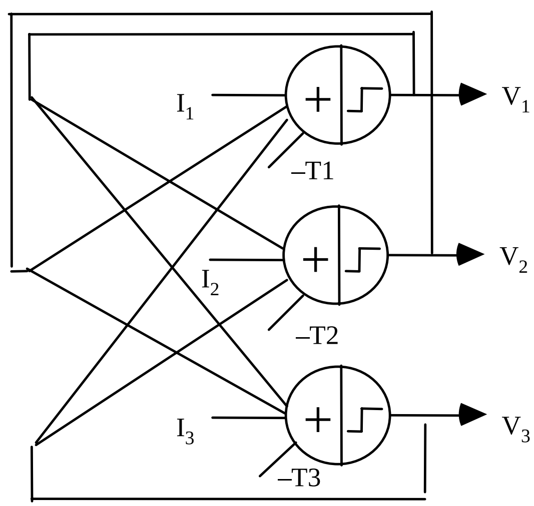
Fundamentals of Hopfield Net

- Proposed by J.J. Hopfield. A fully Connected, feed-back, fixed weight network.
- Each neuron accepts its input from the outputs of all other neurons and the its own input:

Net function
$$u_i = \sum_{i \neq j} w_{ij} v_j + I_i$$
$$I_i = u_i / R_i$$

Output:
$$v_i = \begin{cases} 1 & u_i \geq 0 \\ -1 & u_i < 0. \end{cases}$$



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Discrete Time Formulation

- Define $\underline{V} = [V_1, V_2, \bullet\bullet\bullet, V_n]^T$, $\underline{T} = [T_1, T_2, \bullet\bullet\bullet, T_n]^T$, $\underline{I} = [I_1, I_2, \bullet\bullet\bullet, I_n]^T$, and

$$W = \begin{bmatrix} 0 & w_{12} & w_{13} & \cdots & w_{1n} \\ w_{21} & 0 & w_{23} & \cdots & w_{2n} \\ w_{31} & w_{32} & 0 & \cdots & w_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{n1} & w_{n2} & w_{n3} & \cdots & 0 \end{bmatrix}$$

Then $\underline{V}(t+1) = \mathrm{sgn}\{ W\underline{V}(t) + \underline{I}(t) - \underline{T}(t)\}$

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Example

Let

$$\underline{v}(0)=\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix};\quad W=\begin{bmatrix}0&1&1&-1\\1&0&1&-1\\1&1&0&-1\\-1&-1&-1&0\end{bmatrix};\quad \underline{I}=\underline{T}\equiv\underline{0}$$

Then

$$\underline{v}(1)=\mathrm{sgn}[W\underline{v}(0)]=\mathrm{sgn}\{\begin{bmatrix}1\\1\\3\\-1\end{bmatrix}\}=\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$$

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Example (continued)

$$\underline{v}(2) = \mathrm{sgn}[W\underline{v}(1)] = \mathrm{sgn}\{\begin{bmatrix} 3 \\ 3 \\ 3 \\ -3 \end{bmatrix}\} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \underline{v}(1)$$

[1 1 1 –1]^T and [–1 –1 –1 1]^T are the two stable attractors. Note that

$$W = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

Observations

- Let v* = [1 1 1 1]^T. For any v(0) such that $v^T(0)v^* \neq 0$,

  $$\lim_{t\to\infty} v(t) = v^*$$

  Otherwise, v(t) will oscillate between ±v(0).
- Exercise: try v(0) = [1 1 1 1]^T or [1 1 1 1]^T.
- Discussion:
  - Synchronous update: All neurons are updated together. Suitable for digital implementation
  - Asynchronous update: Some neurons are updated faster than others. Not all neurons are updated simultaneously. Most natural for analog implementation.

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Lyapunov function for Stability

Consider a scalar function E(V) satisfying:

(i) $E(V^*) = 0$

(ii) $E(V) > 0$ for $V \neq V^*$

(iii) $dE/dV = 0$ at $V = V^*$, and $dE/dV < 0$ for $V \neq V^*$

If such an E(V) can be found, it is called a Lyapunov function, and the system is asymptotically stable (i.e. $V \rightarrow V^*$ as $t \rightarrow \infty$).

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Hopfield Net Energy Function

$$E(v) = -\frac{1}{2}\sum_{i \neq j}\sum_{i,j} w_{ij} v_i v_j - \sum_i I_i v_i + \sum_i \frac{1}{R_i}\int_0^{v_i} f_i^{-1}(z)dz$$

$$\nabla_v E(v) = Wv + I - u / R$$

- Hence, Hopfield net dynamic equation is to minimize E(**v**) along descending gradient direction.
- Stability of Hopfield Net – If $w_{ij} = w_{ji}$ & $w_{ii} = 0$, the output will converge to a local minimum (instead of oscillating).

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Associative Retrieval

- Want to store a set of binary input vector $\{b_m; 1 \le m \le M\}$ such that when a perturbed b'_m is presented as I (input), the binary output $V = b_m$.
- <u>*Weight Matrix*</u>: Assume binary values ± 1

$$W = \sum_{m=1}^{M} \left\{ b_m b_m^T - diag\left[b_m b_m^T \right] \right\}$$

$$U = [0 \quad 0 \quad \cdots \quad 0]^T$$

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Example

b_1 = [1 1 1 –1]^T, b_2 = [1 1 –1 –1]^T

$$W = \begin{bmatrix} 0 & 2 & 0 & -2 \\ 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ -2 & -2 & 0 & 0 \end{bmatrix}; \quad U = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Let I = V(0) = [–1 1 –1 –1]^T, then

$$V(1) = f(T \cdot I) = f\left(\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad V(2) = f\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Hopfield Net Solution to TSP

- (Hopfield and Tank) Use an n by n matrix to represent a tour. V_{ij} – i-th city as the j-th stop. Each entry is a neuron!

A	0	1	0	0	0	5
B	0	0	0	1	0	4
C	0	0	0	0	1	3
D	0	0	1	0	0	2
E	1	0	0	0	0	1
City/ tour	1	2	3	4	5	

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Energy Function

$$E = \frac{A}{2}\sum_x \sum_i \sum_{j\neq i} v_{x,i} v_{x,j} + \frac{B}{2}\sum_i \sum_x \sum_{y\neq x} v_{x,i} v_{y,i}$$
$$+\frac{C}{2}\left(\sum_x \sum_i v_{x,i} - N\right)^2 + \frac{D}{2}\sum_x \sum_{x\neq y} \sum_i d_{x,y} v_{x,i}\left(v_{y,i+1} + v_{y,i-1}\right)$$

First three terms makes V a permutation matrix. Last term minimizes the tour distance

$$W_{xi,yj} = -A\delta_{xy}(1-\delta_{ij}) - B\delta_{ij}(1-\delta_{xy}) - C - D\delta_{xy}\left(\delta_{j,i+1} + \delta_{j,i-1}\right)$$

where $\delta_{ij} = 1$ if $i = j$; $= 0$ otherwise.

Validity of the solution – e.g. the A, B, C, D coefficients in the TSP problem.

Quality of the solution – the initial condition will affect the