Interconnect Delay Models
Basic Circuit Analysis Techniques

Network structures & state $\rightarrow$ Natural response $v_N(t)$
(zero-input response)

Input waveform & zero-states $\rightarrow$ Forced response $v_F(t)$
(zero-state response)

For linear circuits: $v(t) = v_N(t) + v_F(t)$

- Output response

- Basic waveforms
  - Step input
  - Pulse input
  - Impulse Input

- Use simple input waveforms to understand the impact of network design
Basic Input Waveforms

unit step function

\[ u(t) = \begin{cases} 
0 & t < 0 \\
1 & t \geq 0 
\end{cases} \]

pulse function of width \( T \)

\[ P_T(t) = \frac{1}{T} \left[ u(t + \frac{T}{2}) - u(t - \frac{T}{2}) \right] \]

unit impulse function

\[ \delta(t) \rightarrow P_T(t) \quad \text{when} \ T \rightarrow 0 \]
\[ \delta(t) = 0 \quad \text{for} \ t \neq 0 \]
\[ \delta(t) \text{ is singular for} \ t = 0 \]

s.t. for any \( \zeta > 0 \)

\[ \int_{-\infty}^{\zeta} \delta(t) \, dt = 1 \]

By definition

\[ u(t) = \int_{-\infty}^{t} \delta(x) \, dx \]

or

\[ \delta(t) = \frac{du(t)}{dt} \]
Step Response vs. Impulse Response

- **Definitions:**
  - (unit) step input \( u(t) \) → (unit) step response \( g(t) \)
  - (unit) impulse input \( \delta(t) \) → (unit) impulse response \( h(t) \)

- **Relationship**

\[
\delta(t) = \frac{du(t)}{dt} \quad \rightarrow \quad h(t) = \frac{dg(t)}{dt}
\]

\[
u(t) = \int_{-\infty}^{t} \delta(x)dx \quad \rightarrow \quad g(t) = \int_{0}^{t} h(x)dx
\]

- **Elmore delay**

\[
T_D = \int_{0}^{\infty} g'(t)t \cdot dt = \int_{0}^{\infty} h(t)t \cdot dt
\]
Analysis of Simple RC Circuit

\[ R \cdot i(t) + v(t) = v_T(t) \]

\[ i(t) = \frac{d}{dt} (Cv(t)) = C \frac{dv(t)}{dt} \]

\[ \Rightarrow RC \frac{dv(t)}{dt} + v(t) = v_T(t) \]

Input waveform

State variable

first-order linear differential equation with constant coefficients
Analysis of Simple RC Circuit

zero-input response: \[ RC \frac{dv(t)}{dt} + v(t) = 0 \]

(natural response) \[ \frac{1}{v(t)} \frac{dv(t)}{dt} = -\frac{1}{RC} \Rightarrow v_N(t) = Ke^{-t/RC} \]

step-input response: \[ RC \frac{dv(t)}{dt} + v(t) = v_0 u(t) \]
\[ v_F(t) = v_0 u(t) \Rightarrow v(t) = Ke^{-t/RC} + v_0 u(t) \]

match initial state: \[ v(0) = 0 \Rightarrow K + v_0 u(t) = 0 \]

output response for step-input: \[ v(t) = v_0 (1 - e^{-t/RC})u(t) \]
Delays of Simple RC Circuit

- $v(t) = v_0(1 - e^{-t/RC})$ under step input $v_0u(t)$

- $v(t) = 0.9v_0 \Rightarrow t = 2.3RC$
  
  $v(t) = 0.5v_0 \Rightarrow t = 0.7RC$

- Commonly used metric
  
  $T_D = RC$ (Elmore delay to be defined later)
Lumped Capacitance Delay Model

- \( R = \) driver resistance
- \( C = \) total interconnect capacitance + loading capacitance
- Sink Delay: \( t_d = R \cdot C \)

- 50% delay under step input = 0.7RC
- Valid when driver resistance >> interconnect resistance
- All sinks have equal delay
Lumped RC Delay Model

\[ t_D = R_d \cdot C_{load} = R_d \cdot (C_{int} + C_g) = R_d \cdot (C_0 \cdot L + C_g) \]

- Minimize delay \( \iff \) minimize wire length
Delay of Distributed RC Lines

\[ V_{out}(t) \xrightarrow{\text{Laplace Transform}} V_{out}(s) \]

\[ V_{out}(s) = \frac{1}{s \cosh \sqrt{sRC}} \]

\[ \cosh(x) = \frac{e^x + e^{-x}}{2} \]

\[ = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \]

Step response of distributed and lumped RC networks. A potential step is applied at \( V_{IN} \), and the resulting \( V_{OUT} \) is plotted. The time delays between commonly used reference points in the output potential is also tabulated.
Delay of Distributed RC Lines (cont’d)

<table>
<thead>
<tr>
<th>Output potential range</th>
<th>Time elapsed (Distributed RC Network)</th>
<th>Time elapsed (Lumped RC Network)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to 90%</td>
<td>1.0 $RC$</td>
<td>2.3 $RC$</td>
</tr>
<tr>
<td>10% to 90% (rise time)</td>
<td>0.9 $RC$</td>
<td>2.2 $RC$</td>
</tr>
<tr>
<td>0 to 63%</td>
<td>0.5 $RC$</td>
<td>1.0 $RC$</td>
</tr>
<tr>
<td>0 to 50% (delay)</td>
<td>0.4 $RC$</td>
<td>0.7 $RC$</td>
</tr>
<tr>
<td>0 to 10%</td>
<td>0.1 $RC$</td>
<td>0.1 $RC$</td>
</tr>
</tbody>
</table>
Distributed Interconnect Models

- Distributed RC circuit model
  - L,T or Π circuits

- Distributed RCL circuit model

- Tree of transmission lines
Distributed RC Circuit Models
Distributed RLC Circuit Model
Delays of Complex Circuits under Unit Step Input

- Circuits with monotonic response
- Easy to define delay & rise/fall time
- Commonly used definitions
  - Delay $T_{50\%} = \text{time to reach half-value, } v(T_{50\%}) = 0.5V_{dd}$
  - Rise/fall time $T_R = 1/v'(T_{50\%})$ where $v'(t)$: rate of change of $v(t)$ w.r.t. $t$
  - Or rise time = time from 10% to 90% of final value
- Problem: lack of general analytical formula for $T_{50\%}$ & $T_R$!
Delays of Complex Circuits under Unit Step Input (cont’d)

- Circuits with non-monotonic response

- Much more difficult to define delay & rise/fall time
Elmore Delay for Monotonic Responses

• Assumptions:
  – Unit step input
  – Monotone output response

• Basic idea: use of mean of \( v'(t) \) to approximate median of \( v'(t) \)

\( v(t) \): output response (monotone)

\( v'(t) \): rate of change of \( v(t) \)
Elmore Delay for Monotonic Responses

- $T_{50\%}$: median of $v'(t)$, since

$$
\int_0^{T_{50\%}} v'(t) dt = \int_{T_{50\%}}^{+\infty} v'(t) dt = \text{half of final value of } v(t) \quad \text{(by def.)}
$$

- Elmore delay $T_D = \text{mean of } v'(t)$

$$
T_D = \int_0^{+\infty} v'(t) t dt
$$
Why Elmore Delay?

• Elmore delay is easier to compute analytically in most cases
  – Verified later on by many other researchers, e.g.
    • Elmore delay for RC trees [Penfield-Rubinstein, DAC’81]
    • Elmore delay for RC networks with ramp input [Chan, T-CAS’86]
    • ..... 

• For RC trees: [Krauter-Tatuianu-Willis-Pileggi, DAC’95]
  \[T_{50\%} \leq T_D\]

• Note: Elmore delay is not 50% value delay in general!
Elmore Delay for RC Trees

• Definition
  – $h(t)$ = impulse response
  – $T_D$ = mean of $h(t)$
    $$= \int_0^\infty h(t) \cdot t \, dt$$

• Interpretation
  – $H(t)$ = output response (step process)
  – $h(t)$ = rate of change of $H(t)$
  – $T_{50\%}$ = median of $h(t)$
  – Elmore delay approximates the median of $h(t)$ by the mean of $h(t)$
Elmore Delay of a RC Tree

[Rubinstein-Penfield-Horowitz, T-CAD’83]

**Lemma:** when a step input is applied to a RC tree

\[ v_i(t) \text{ is monotonic in } t \text{ for every node } i \text{ in tree} \]

**Proof:**

\[ \iff v'_i(t) \geq 0 \text{ at every node } i \quad (v'_i(t) = h_i(t)) \]

\[ \iff \text{impulse response } h_i(t) \geq 0 \text{ at every node } i \]

Let \( h_{\text{min}}(t) \) be the min. voltage of any node at \( t \)

\[ h_{\text{min}}(0+) \geq 0 \]

Assume that \( h_{\text{min}}(t_0) < 0 \)

Then, \( \exists t_1 < t_0 \quad \text{s.t.} \quad h'_{\text{min}}(t_1) < 0 \)

Let node \( i_{\text{min}} \) achieve \( h_{\text{min}}(t_1) \) at \( t_1 \)

Then, the current from any node \( i \) to \( i_{\text{min}} \) is \( \geq 0 \) at \( t_1 \)

Since \( h_i(t_1) \geq h_{\text{min}}(t_1) \) & \( i \) connects \( i_{\text{min}} \) via resistors

Since all currents \( i \rightarrow i_{\text{min}} \) charge the capacitor at \( i_{\text{min}} \)

\[ h'_{\text{min}}(t_1) \geq 0 \quad \Rightarrow \quad \text{contradiction!} \]
Elmore Delay in a RC Tree (cont’d)

\( P_i \) : path from input to node \( i \); \( S_i \) subtree rooted at \( i \)

\( R_{jk} \) : resistance of common path

\( P_j \cap P_k \) from input to \( j \) & \( k \)

**Theorem** : Elmore delay to node \( i \)

\[ T_{D_i} = \sum_k R_{ki} C_k \]

**Proof** : The current to cap. of node \( i = C_i \frac{dv_i(t)}{dt} \)

\( 1 - v_i(t) \) = The voltage drop on \( P_i = \sum_{k \in P_i} R_k \cdot \) (current to all cap’s in \( S_i \))

\[ = \sum_k \text{(current to cap } k \text{)} \cdot \text{(common path res. between } P_i \text{ and } P_k \text{)} \]

\[ = \sum_k C_k \frac{dv_k(t)}{dt} \cdot R_{ki} = \sum_k R_{ki} C_k \frac{dv_k(t)}{dt} \]

\[ T_{D_i} = \int_0^\infty v_i'(t) \cdot dt = v_i(t) \cdot t \bigg|_0^\infty - \int_0^\infty v_i(t) dt \]

\[ = \lim_{T \to \infty} [v_i(T) \cdot T - \int_0^T v_i(t) dt] = \lim_{T \to \infty} (v_i(T) - 1) \cdot T + \int_0^\infty (1 - v_i(t)) dt \]
Elmore Delay in a RC Tree (cont’d)

- We shall show later on that \( \lim_{T \to \infty} (1 - v_i(T)) \cdot T = 0 \)
  i.e. \( 1 - v_i(T) \) goes to 0 at a much faster rate than \( 1/T \) when \( T \to \infty \)

- Let \( f_i(t) = \int_0^t [1 - v_i(x)] \, dx \)
  \[
  f_i(t) = \int_0^t \sum_k R_{ki} C_k \frac{dv_k(x)}{dx} \, dx \\
  = \sum_k R_{ki} C_k v_k(t) \\
  = \sum_k R_{ki} C_k - \sum_k R_{ki} C_k [1 - v_k(t)] \\
  f_i(\infty) = \sum_k R_{ki} C_k \\
  \therefore \ T_{D_i} = \lim_{T \to \infty} (1 - v_i(T)) T + \int_0^\infty [1 - v_i(t)] \, dt \\
  = f_i(\infty) = \sum_k R_{ki} C_k \]
Some Definitions For Signal Bound Computation

Let \( T_p = \sum_k R_{kk} C_k \)

\[ T_{R_i} = \left( \sum_k R_{ki}^2 C_k \right) / R_{ii} \]

Then, \( T_{R_i} \leq T_{D_i} \leq T_p \)

Recall \( T_{D_i} = \sum_k R_{ki} C_k \)

(since \( R_{kk} \geq R_{ki} \) & \( R_{ii} \geq R_{ki} \))
Signal Bounds in RC Trees

- Theorem

Lower bounds

\[
v_i(t) \geq \begin{cases} 
0 & t \geq 0 \\
1 - \frac{T_{D_i}}{t + T_{R_i}} & t \geq 0 \text{ (non-trivial when } t \geq T_{D_i} - T_{R_i}) \\
1 - \frac{T_{D_i}}{T_p} \left(\frac{t - t}{T_p}\right) & t \geq T_p - T_{R_i} \\
1 - \frac{T_{R_i}}{T_p} \left(\frac{t - t}{T_{R_i}}\right) & t \geq T_p - T_{R_i} 
\end{cases}
\]

Upper bounds

\[
v_i(t) \leq \begin{cases} 
1 - \frac{T_{D_i} - t}{T_p} & t \geq 0 \\
1 - \frac{T_{R_i}}{T_p} \left(\frac{t - t}{T_{R_i}}\right) & t \geq T_{D_i} - T_{R_i} \\
1 - \frac{T_{R_i}}{T_p} \left(\frac{t - t}{T_{R_i}}\right) & t \geq T_{D_i} - T_{R_i} 
\end{cases}
\]
Proofs of Signal Bounds in RC Trees

• **Lemma:** \[ R_{ii} [1 - \nu_{k} (t)] \geq R_{ki} [1 - \nu_{i} (t)] \] \hspace{1cm} (2)

**Proof:**
\[ R_{ii} \geq \max(R_{ki}, R_{ji}) \quad R_{jk} \geq \min(R_{ki}, R_{ji}) \]
\[ : \quad R_{ii} \cdot R_{jk} \geq R_{ki} \cdot R_{ji} \]
\[ R_{ii} [1 - \nu_{k} (t)] - R_{ki} [1 - \nu_{i} (t)] \]
\[ = R_{ii} \sum_{j} R_{jk} C_{j} \frac{dv_{j} (t)}{dt} - R_{ki} \sum_{j} R_{ji} C_{j} \frac{dv_{j} (t)}{dt} \]
\[ = \sum_{j} (R_{ii} R_{jk} - R_{ki} R_{ji}) C_{j} \frac{dv_{j} (t)}{dt} \geq 0 \]
(Since \( v_{j} (t) \) is monotonic)

• **Lemma:** \[ R_{ki} [1 - \nu_{k} (t)] \leq R_{kk} [1 - \nu_{i} (t)] \] \hspace{1cm} (3)
Proofs of Signal Lower Bounds in RC Trees

From (1):
\[ T_{D_i} - f_i(t) = \sum_k R_{ki} C_k (1 - v_k(t)) \]

From (2) & (3):
\[ T_{R_i} [1 - v_i(t)] \leq T_{D_i} - f_i(t) \leq T_p [1 - v_i(t)] \]  \hspace{1cm} (4)

i.e.
\[ \frac{T_{D_i} - f_i(t)}{T_p} \leq 1 - v_i(t) = \frac{df_i(t)}{dt} \leq \frac{T_{D_i} - f_i(t)}{T_{R_i}} \]

Thus,
\[ \frac{1}{T_p} \leq \frac{1}{T_{D_i} - f_i(t)} \cdot \frac{df_i(t)}{dt} \leq \frac{1}{T_{R_i}} \]

Integrating from \( t_1 \) to \( t_2 \):
\[ \frac{(t_2 - t_1)}{T_p} \leq -\ln(T_{D_i} - f_i(t)) \bigg|_{t_1}^{t_2} \leq \frac{(t_2 - t_1)}{T_{R_i}} \]

i.e.
\[ e^{-(t_2-t_1)/T_p} \geq \frac{T_{D_i} - f_i(t_2)}{T_{D_i} - f_i(t_1)} \geq e^{-(t_2-t_1)/T_{R_i}} \]  \hspace{1cm} (5)

Also, since \( v_i(t) \) is monotonic
\[ (t_4 - t_3)[1 - v_i(t_4)] \leq f_i(t_4) - f_i(t_3) \]  \hspace{1cm} (6)

Let \( t_3 = 0, \ t_4 = t \)
\[ f_i(t) \geq [1 - v_i(t)] \cdot t \]  \hspace{1cm} (7)
Proofs of Signal Lower Bounds in RC Trees

From left - half of (4) and (7)

\[ T_{R_i} [1 - v_i(t)] \leq T_{D_i} - f_i(t) \leq T_{D_i} - [1 - v_i(t)] t \]

\[ \Rightarrow 1 - v_i(t) \leq \frac{T_{D_i}}{t + T_{R_i}} \quad \therefore v_i(t) \geq 1 - \frac{T_{D_i}}{t + T_{R_i}} \]

Let \( t_3 = t - T_p + T_{R_i} \quad t_4 = t \) in (6)

\[ (T_p - T_{R_i})[1 - v_i(t)] \leq f_i(t) - f_i(t_3) \quad (8) \]

Let \( t_1 = 0 \) and \( t_2 = t_3 \) in left - half of (5)

\[ T_{D_i} - f_i(t_3) \leq T_{D_i} e^{-\frac{t_3}{T_p}} \quad (9) \]

From left - half of (4)

\[ T_{R_i} [1 - v_i(t)] \leq T_{D_i} - f_i(t) \quad (10) \]

(8) + (9) + (10):

\[ T_p [1 - v_i(t)] \leq T_{D_i} e^{-\frac{t_3}{T_p}} \]

\[ \therefore v_i(t) \geq 1 - \frac{T_{D_i}}{T_p} e^{-\frac{t_3}{T_p}} = 1 - \frac{T_{D_i}}{T_p} e^{-\frac{(T_p - T_{R_i})}{T_p}} \cdot e^{-t/T_p} \]
Delay Bounds in RC Trees

Lower bounds:
\[ t \geq T_{D_i} - T_p \left[ 1 - v_i(t) \right] \]
\[ t \geq T_{D_i} - T_{R_i} + T_{R_i} \ln \frac{T_{R_i}}{T_p \left[ 1 - v_i(t) \right]} \]  
when \( v_i(t) \geq 1 - \frac{T_{R_i}}{T_p} \)

Upper bounds:
\[ t \leq \frac{T_{D_i}}{1 - v_i(t)} - T_{R_i} \]
\[ t \leq T_p - T_{R_i} + T_p \ln \frac{T_{D_i}}{T_p \left[ 1 - v_i(t) \right]} \]  
when \( v_i(t) \geq 1 - \frac{T_{D_i}}{T_p} \)
Computation of Elmore Delay & Delay Bounds in RC Trees

• Let $C(T_k)$ be total capacitance of subtree rooted at $k$
• Elmore delay

$$T_{D_i} = \sum_{k \in p_i} R_k \cdot C(T_k)$$

upper bound:

$$T_p = \sum_k R_k \cdot C(T_k)$$

lower bound:

$$T_{R_i} = \sum_k R_{ki}^2 \cdot \frac{C_k}{R_{ii}}$$

*all three formula can be computed in linear time recursively in a bottom-up fashion*
Comments on Elmore Delay Model

• Advantages
  – Simple closed-form expression
    • Useful for interconnect optimization
  – Upper bound of 50% delay [Gupta et al., DAC’95, TCAD’97]
    • Actual delay asymptotically approaches Elmore delay as input signal rise time increases
  – High fidelity [Boese et al., ICCD’93],[Cong-He, TODAES’96]
    • Good solutions under Elmore delay are good solutions under actual (SPICE) delay
Comments on Elmore Delay Model

• Disadvantages
  – Low accuracy, especially poor for slope computation
  – Inherently cannot handle inductance effect
    • Elmore delay is first moment of impulse response
    • Need higher order moments