Solving Linear Systems

Last time we learned about how to solve systems $Ax = b$, when $A$ was symmetric and positive-definite.
- The key was to factor the matrix into two triangular matrices $A = LU$.
- In the case that $A$ is SPD, then we can always do this, and in fact $U = L^T$.
- What if $A$ is not SPD?
- The workhorse in this case is the LU-decomposition.
- LU-decomposition is very related to (the well-known) Gaussian elimination, a fact we will try to make clear today...

Gaussian Elimination

An example for today. Let's solve it...

\[
\begin{align*}
    x_1 + x_2 + 2x_3 &= 3 \\
    2x_1 + 3x_2 + x_3 &= 2 \\
    3x_1 - x_2 - x_3 &= 6
\end{align*}
\]

- Subtract twice first equation from the second
- Subtract 3 times the first equation from the third
- Then add 4 times second equation to the third
- You've made a triangular system!
- What were the matrices that produce this?

Elementary Dear Watson!

- We reduced the columns by taking linear combinations of the rows of the matrix.
- This implies that the reduction process can be thought of as a multiplication of $A$ on the left by some matrix.
- What does the matrix look like?
- It is an elementary matrix of the form $E = I - uv^T$.
- In fact, it’s a special form of an elementary matrix: It will be a unit lower triangular matrix with multipliers only in one column.
The Elimination Matrix

Let's find a matrix $M_1$ that reduces the first column of $A$.

$$M_1 = \begin{pmatrix} 1 & m_{21} & 1 & m_{31} & 1 & \ldots & m_{n1} & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & 0 & 0 & \ldots & 0 & \end{pmatrix}$$

By our properties of matrix multiplication, this matrix

- Leaves the first row of $A$ alone
- Takes $-m_{21}$ times the first row, adds the second row, and puts this in the second row of the new matrix $M_1 A$
- Takes $-m_{31}$ times the first row of $A$, adds the third row, and puts this in the third row of the new matrix $M_1 A$
- (And So On...)

The UpShot

- To eliminate the first column, we want
  $$m_{i1} = \frac{a_{i1}}{a_{11}}$$
- Note: These were exactly the multipliers we used in our simple example
- $a_{11}$ is called the pivot element, and this reduction only works if the pivot element is $\neq 0$
- Next time: What happens if pivot element is 0 (or small)

Let's Carry On

$$M_1 A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \ldots & a_{nn} \\ \end{pmatrix}$$

To reduce the second column, we would like a matrix

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 1 & 0 & \ldots & 0 & 1 \\ 0 & -m_{32} & 1 & \ldots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -m_{n2} & 0 & \ldots & 1 \\ \end{pmatrix}$$

Matrix Effect

Again, the matrix $M_2$ will...

- Leave first row of $M_1 A$ unchanged in $M_2 M_1 A$
- Leave second row of $M_1 A$ unchanged $M_2 M_1 A$
- Take $-m_{32}$ times second row + third row in $M_2 M_1 A$

Lather Rinse Repeat

- Repeat this $n - 1$ times
- In the end, we get $M_{n-1} \ldots M_2 M_1 A = U$
- Fact: The product of unit lower triangular matrices is unit lower triangular
- So in the end we have $M A = U$, with $M$ unit lower triangular, and $U$ upper triangular
- This process is known as Gaussian Elimination, and the matrix $M$ is known as the product form of the LU factorization
Finding LU Directly

- Here is a recursive method for finding the LU factorization.
- We'll divide the matrix $A$ into four pieces:

  \[
  A = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
  \end{bmatrix}
  \]

  \[
  = \begin{bmatrix}
  a_{11} & w^T \\
  v & A'
  \end{bmatrix}
  \]

  (1)

- Next, we'll use row operations to change $v$ into the zero vector and record the operations in another matrix.

Finding the LU Decomposition (cont.)

- By simple multiplication, you can verify the following factorization of $A$:

  \[
  A = \begin{bmatrix}
  a_{11} & w^T \\
  v & A'
  \end{bmatrix}
  \]

  (3)

  \[
  = \begin{bmatrix}
  1 & 0 \\
  v/a_{11} & I
  \end{bmatrix} \begin{bmatrix}
  a_{11} & w^T \\
  0 & A' - vv^T/a_{11}
  \end{bmatrix}
  \]

  (4)

- We can show that if $A$ is nonsingular, then so is $A' - vv^T/a_{11}$.
- So we can recursively call the method to factor the $(n-1) \times (n-1)$ matrix $A' - vv^T/a_{11}$.
- Applying this recursion $n$ times yields the desired factorization.

The Algorithm

**LU-DECOMPOSITION**

1. $n \leftarrow \text{rows}[L]$
2. for $k \leftarrow 1$ to $n$
3. do
4. \quad $u_{kk} \leftarrow a_{kk}$
5. \quad for $i \leftarrow 1$ to $n$
6. \quad do
7. \quad \quad $\ell_{ik} \leftarrow a_{ik}/u_{kk}$
8. \quad \quad $u_{ki} \leftarrow a_{ki}$
9. \quad \quad for $i \leftarrow k+1$ to $n$
10. \quad \quad do
11. \quad \quad \quad $a_{ij} \leftarrow a_{ij} - \ell_{ik}u_{kj}$
LU $\approx$ Gaussian Elimination

- We either have $A = LU$ or we have $MA = U$
- $M$ is unit lower triangular, and in fact the inverse of a unit lower triangular matrix is unit lower-triangular, so $A = M^{-1}U$, and since the elements of $L$ and $U$ are unique, it must be that $L = M^{-1}$
- Because of the special structure of $M$, we have a (fairly) remarkable relationship

The relationship is the following:

$$M^{-1} = (M_{n-1} \cdots M_2 M_1)^{-1} = M_1^{-1} M_2^{-1} \cdots = L$$

$$L = \begin{pmatrix}
1 & 1 \\
\frac{m_{21}}{m_{11}} & m_{21} & 1 \\
\vdots & \ddots & \ddots \\
\frac{m_{n1}}{m_{11}} & \frac{m_{n2}}{m_{11}} & \frac{m_{n3}}{m_{11}} & \cdots & 1
\end{pmatrix}$$

where the $m_{ik}$ are the multipliers from Gaussian elimination!
- So $L$ and $U$ can be derived directly from the elimination process:

$$\ell_{ik} = m_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}$$
$$u_{kj} = a_{kj}^{(k)}$$