Valid Inequalities for the Knapsack Problem

- We are interested in valid inequalities for the knapsack set $\text{KNAP}$

$$\text{KNAP} = \{x \in \mathbb{B}^n \mid \sum_{j \in N} a_j x_j \leq b\}$$

- $N = \{1, 2, \ldots, n\}$
- We assume WLOG that $a_j > 0 \ \forall j \in N$ Why?
  - If $a_j < 0$, let $\hat{x}_j = 1 - x_j$
- We will also assume that $a_j < b \ \forall j \in N$
- We are interested in finding facets of $\text{conv}(\text{KNAP})$

Using Valid Inequalities for a Relaxation

- I want to solve MIPs, why do I care about strong inequalities for the knapsack problem?
- If $P = \{x \in \mathbb{B}^n \mid Ax \leq b\}$, then for any row $i$, $P_i = \{x \in \mathbb{B}^n \mid a_i^T x \leq b_i\}$ is a relaxation of $P$.
  - $P \subseteq P_i \ \forall i = 1, 2, \ldots, m$
  - $P \subseteq \bigcap_{i=1}^m P_i$
- Any inequality valid for a relaxation of an IP is valid for the IP itself.
- Generating valid inequalities for a relaxation is often easier.
- If the intersection of the relaxations is a good approximation to the true problem, then the inequalities will be quite useful.
- Crowder, Johnson, and Padberg is the seminal paper that shows this to be true.
Simple facets

- What is \( \text{dim}(\text{conv}(\text{KNAP})) \)?
  - \( 0, e_j, \forall j \in N \) are \( n + 1 \) affinely independent points in \( \text{conv}(\text{KNAP}) \) \( \Rightarrow \) \( \text{dim}(\text{conv}(\text{KNAP})) = n \).
- \( x_k \geq 0 \) is a facet of \( \text{conv}(\text{KNAP}) \)
  - \textbf{Proof}. \( 0, e_j, \forall j \in N \setminus k \) are \( n \) affinely independent points that satisfy \( x_k = 0 \).
- \( x_k \leq 1 \) is a facet of \( \text{conv}(\text{KNAP}) \) if \( a_j + a_k \leq b \forall j \in N \setminus k \)
  - \textbf{Proof}. \( e_k, e_j + e_k, \forall j \in N \setminus k \) are \( n \) affinely independent points that satisfy \( x_k = 1 \).

Covers

- A set \( C \subseteq N \) is a cover if \( \sum_{j \in C} a_j > b \)
- A cover \( C \) is a minimal cover if \( C \setminus j \) is not a cover \( \forall j \in C \)
- If \( C \subseteq N \) is a cover, then the cover inequality
  \[
  \sum_{j \in C} x_j \leq |C| - 1
  \]
  is a valid inequality for \( S \)

Example

\[
\text{MYKNAP} = \{ x \in \mathbb{B}^7 \mid 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19 \}
\]

- Some minimal covers are the following:
  - \( x_1 + x_2 + x_3 \leq 2 \)
  - \( x_1 + x_2 + x_6 \leq 2 \)
  - \( x_1 + x_5 + x_6 \leq 2 \)
  - \( x_3 + x_4 + x_5 + x_6 \leq 3 \)

Best We Can Do?

- Are these inequalities the strongest ones we can come up with?
  - What does strongest mean?
    - We all know that facets are the “strongest”, but can we say anything else?
  - If \( \pi^T x \leq \pi_0 \) and \( \mu^T x \leq \mu_0 \) are two valid inequalities for \( P \subseteq \mathbb{R}_+^n \), we say that \( \pi^T x \leq \pi_0 \) dominates \( \mu^T x \leq \mu_0 \) if \( \exists u \geq 0 \) such that
    - \( \pi \geq u\mu \)
    - \( \pi_0 \leq u\mu_0 \)
    - \( (\pi, \pi_0) \neq u(\mu, \mu_0) \)
  - If \( \pi^T x \leq \pi_0 \) dominates \( \mu^T x \leq \mu_0 \), then
    \( \{ x \in \mathbb{R}_+^n \mid \pi^T x \leq \pi_0 \} \subseteq \{ x \in \mathbb{R}_+^n \mid \mu^T x \leq \mu_0 \} \)
Back to the Knapsack

- If \( C \subseteq N \) is a minimal cover, the extended cover \( E(C) \) is defined as:
  - \( E(C) = C \cup \{ j \in N \mid a_j = a_i \forall i \in C \} \)
- If \( E(C) \) is an extended cover for \( S \), then the extended cover inequality
  \[
  \sum_{j \in E(C)} x_j \leq |C| - 1,
  \]
is a valid inequality for \( S \).
- **Proof.** \( x^R \in \text{Knap}, \sum_{j \in E(C)} x^R_j \geq |C| \Rightarrow |R \cap E(C)| \geq |C| \).
- \( \sum_{j \in R} a_j \geq \sum_{j \in R \cap E(C)} a_j \geq \sum_{j \in E(C)} a_j > b \), so \( x^R \notin \text{Knap} \).
- Note this inequality dominates the cover inequality if \( E(C) \setminus C \neq \emptyset \).
- **(Example, cont.)** The cover inequality \( x_3 + x_4 + x_5 + x_6 \leq 3 \) is dominated by the extended cover inequality \( x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3 \).

As Good As It Gets?

- **Question:** Are these inequalities as strong as possible?
- **Answer:** Sometimes.
- Order the variables so that \( a_1 \geq a_2 \ldots \geq a_n \).
- Denote the cover as \( C = \{ j_1, j_2, \ldots, j_r \} \) \( (j_1 < j_2 < \ldots < j_r) \) so that \( a_{j_1} \geq a_{j_2} \geq \ldots \geq a_{j_r} \).
- Given a minimal cover \( C \), define the following sets:
  - \( R_k = C \setminus k \ \forall k \in C \)
  - \( S_k = C \setminus \{ j_1, j_2 \} \cup \{ k \} \ \forall k \in E(C) \setminus C \)
  - \( T_k = C \setminus j_1 \cup \{ k \} \ \forall k \in N \setminus E(C) \)

Facets

- If \( C = N \), \( \sum_{j \in C} x_j \leq |C| - 1 \) is a facet of \( \text{conv(Knap)} \).
  - **Proof.** \( R_k \).
- Let \( p = \min\{ j \mid j \in N \setminus E(C) \} \). If \( C = E(C) \), and \( \sum_{j \in C \setminus j_1} a_j + a_p \leq b \), then \( \sum_{j \in C} x_j \leq |C| - 1 \) is a facet of \( \text{conv(Knap)} \).
  - **Proof.** \( R_k \) and \( T_k \).
- If \( E(C) = N \) and \( \sum_{j \in E(C) \setminus \{ j_1, j_2 \}} a_j + a_1 \leq b \), then \( \sum_{j \in E(C)} x_j \leq |C| - 1 \) is a facet of \( \text{conv(Knap)} \).
  - **Proof.** \( R_k \) and \( S_k \).

In General...

- Order the variables so that \( a_1 \geq a_2 \ldots \geq a_n \).
- Let \( C \) be a cover with \( C = \{ j_1, j_2, \ldots, j_r \} \) \( (j_1 < j_2 < \ldots < j_r) \) so that \( a_{j_1} \geq a_{j_2} \geq \ldots \geq a_{j_r} \). Let \( p = \min\{ j \mid j \in N \setminus E(C) \} \).
- If any of the following conditions hold, then
  \[
  \sum_{j \in E(C)} x_j \leq |C| - 1
  \]
gives a facet of \( \text{conv(Knap)} \)
  - \( C = N \)
    - \( E(C) = N \) and \( (*) \sum_{j \in C \setminus \{ j_1, j_2 \}} a_j + a_1 \leq b \)
    - \( C = E(C) \) and \( (**) \sum_{j \in C \setminus j_1} a_j + a_p \leq b \)
    - \( C \subset E(C) \subset N \) and \( (*) \) and \( (**) \).
Covers and Lifting

- Let $P_{1,2,7} = \text{conv}(\text{MYKNAP} \cap \{x \in \mathbb{R}^7 \mid x_1 = x_2 = x_7 = 0\})$
- Consider the cover inequality arising from $C = \{3, 4, 5, 6\}$.
- $\sum_{j \in C} x_j \leq 3$ is facet defining for $P_{1,2,7}$
- If $x_1$ is not fixed at 0, can we strengthen the inequality?
- For what values of $\alpha_1$ is the inequality $\alpha_1 x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$

valid for

$P_{2,7} = \text{conv}(\{x \in \text{MYKNAP} \mid x_2 = x_7 = 0\})$

- If $x_1 = 0$ then the inequality is valid for all values of $\alpha_1$

The Other Case

- If $x_1 = 1$, the inequality is valid if and only if

$$\alpha_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

is valid for all $x \in \mathbb{R}^4$ satisfying

$$6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 - 11$$

- Equivalently, if and only if

$$\alpha_1 + \max_{x \in \mathbb{R}^4} \{x_3 + x_4 + x_5 + x_6 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8\} \leq 3$$

- Equivalently if and only if $\alpha_1 \leq 3 - \gamma$, where

$$\gamma = \max_{x \in \mathbb{R}^4} \{x_3 + x_4 + x_5 + x_6 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8\}$.
Solving the Knapsack Problem

- In this case, we can “solve” the knapsack problem to see that $\gamma = 1$. Therefore $\alpha_1 \leq 2$.
- The inequality

$$2x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

is a valid inequality for $P_{27}$.
- Is it facet-defining?

Lifting

- What we’ve just done is called lifting. Where a valid (and facet defining) inequality for $S \cap \{x \in \mathbb{B}^n \ | \ x_k = 0\}$ is turned into a facet defining inequality for $S$.
- **Theorem.** Let $S \subseteq B^n$, for $\delta \in \{0, 1\}$, $S^\delta = S \cap \{x \in \mathbb{B}^n \ | \ x_1 = \delta\}$. Suppose

$$\sum_{j=2}^{n} \pi_j x_j \leq \pi_0$$

is valid for $S^0$.

Lifting Thm. (2)

- If $S^1 = \emptyset$, then $x_1 \leq 0$ is valid for $S$
- If $S^1 \neq \emptyset$, then

$$\alpha_1 x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0$$

is valid for $S$ for any $\alpha_1 \leq \pi_0 - \gamma$, where

$$\gamma = \max\{\sum_{j=2}^{n} \pi_j x_j \ | \ x \in S^1\}.$$
For You To Do...

- Read N&W Sections II.2.1, II.2.2
- Read [2]
- Read [1]
