An “Uplifting” Experience

- \( S \subseteq \mathbb{B}^n \)
- Lifting is a process in which a valid (and facet defining) inequality for \( S \cap \{ x \in \mathbb{B}^n \mid x_k = 0 \} \) is turned into a facet defining inequality for \( S \).
- **Theorem.** Let \( S \subseteq \mathbb{B}^n \), for \( \delta \in \{ 0, 1 \} \), \( S^\delta = S \cap \{ x \in \mathbb{B}^n \mid x_1 = \delta \} \). Suppose
  \[
  \sum_{j=2}^{n} \pi_j x_j \leq \pi_0
  \]
  is valid for \( S^0 \).

Lifting Thm. (2)

- If \( S^1 = \emptyset \), then \( x_1 \leq 0 \) is valid for \( S \)
- If \( S^1 \neq \emptyset \), then \( \alpha_1 x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0 \) is valid for \( S \) for any \( \alpha_1 \leq \pi_0 - \gamma \), where
  \[
  \gamma - \max \{ \sum_{j=2}^{n} \pi_j x_j \mid x \in S^1 \}.
  \]
- If \( \alpha_1 = \pi_0 - \gamma \) and \( \sum_{j=2}^{n} \pi_j x_j \leq \pi_0 \) defines a face of dimension \( k \) of \( \text{conv}(S^0) \), then
  \[
  \alpha_1 x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0
  \]
  defines a face of dimension at least \( k + 1 \) of \( \text{conv}(S) \).
Uplifting Example

- Let $P_{1,2,7} = \text{conv}(\text{MYKNAP} \cap \{x \in \mathbb{R}^7 \mid x_1 = x_2 = x_7 = 0\})$
- Consider the cover inequality arising from $C = \{3, 4, 5, 6\}$.
- $\sum_{j \in C} x_j \leq 3$ is facet defining for $P_{1,2,7}$
- If $x_1$ is not fixed at 0, can we strengthen the inequality?
- For what values of $\alpha_1$ is the inequality
  \[
  \alpha_1 x_1 + x_3 + x_4 + x_5 + x_6 \leq 3
  \]
  valid for
  \[
  P_{2,7} = \text{conv}(\{x \in \text{MYKNAP} \mid x_2 = x_7 = 0\})?
  \]
- If $x_1 = 0$ then the inequality is valid for all values of $\alpha_1$

You Can Also “DownLift”

- $s \subseteq \mathbb{B}^n, S^1 = S \cap \{x \in \mathbb{B}^n \mid x_1 = 1\}$
- Let $\sum_{j=2}^n \pi_j x_j \leq \pi_0$ be valid for $S^1$.
- If $S^0 = \emptyset$, $x_1 \geq 1$ is valid for $S$, otherwise
  \[
  \xi_1 x_1 + \sum_{j=2}^n \pi_j x_j \leq \pi_0 + \xi_1
  \]
  is valid for $S$, for $\xi_i \geq \gamma - \pi_0$
  - $\gamma = \max\{\sum_{j=2}^n \pi_j x_j \mid x \in S^0\}$.
- Similar facet/dimension results to uplifting if the lifting is maximum.

Uplifting Example (2)

- If $x_1 = 1$, the inequality is valid if and only if
  \[
  \alpha_1 + x_3 + x_4 + x_5 + x_6 \leq 3
  \]
  is valid for all $x \in \mathbb{B}^4$ satisfying
  \[
  6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 - 11
  \]
- Equivalently, if and only if
  \[
  \alpha_1 + \max\{x_3 + x_4 + x_5 + x_6 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8\} \leq 3
  \]
- Equivalently if and only if $\alpha_1 \leq 3 - \gamma$, where
  \[
  \gamma = \max\{x_3 + x_4 + x_5 + x_6 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8\}.
  \]

DownLifting Example

- Let $P_6^1 = \text{conv}(\text{MYKNAP} \cap \{x \in \mathbb{R}^7 \mid x_6 = 1\})$
- **Fact:** $x_1 + x_5 \leq 1$ is facet-defining for $P_6^1$.
  - $C = E(C)$ and $\sum_{j \in C \setminus j_5} a_j + a_p \leq b$
  - **Note:** $x_1 + x_5 \leq 1$ is not valid for MYKNAP
- For what values of $\alpha$ is the inequality $x_1 + x_5 + \alpha(x_6 - 1) \leq 1$
  valid for MYKNAP?
- If $x_6 = 1$, then valid if $\alpha \in [-\infty, \infty]$
- If $x_6 = 0$, then valid if $\alpha \geq x_1 + x_5 - 1$ $\forall x \in \text{MYKNAP}$
- If and only if $\alpha \geq \max_{x \in \mathbb{R}^7}\{x_1 + x_5 - 1 \mid 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 7x_7 \leq 19\}$
- $\alpha \geq 1$.
- $x_1 + x_5 + x_6 \leq 2$ is valid and facet-defining inequality for MYKNAP.
General Lifting and SuperAdditivity

- $K = \text{conv}(\{x \in \mathbb{Z}_+^N, y \in \mathbb{R}_+^M \mid a^T x + g^T y \leq b, x \leq u\})$
- Partition $N$ into $[L, U, R]$
  - $L = \{i \in N \mid x_i = 0\}$
  - $U = \{i \in N \mid x_i = u_i\}$
  - $R = N \setminus L \setminus U$
- We will use the notation: $x_R$ to mean the vector of variables that are in the set $R$.
  - $a^T_R x_R = \sum_{j \in R} a_j x_j$

$L(K, U) = \text{conv}(\{x \in \mathbb{Z}_+^N, y \in \mathbb{R}_+^M \mid a^T_R x_R + g^T y \leq d, x_R \leq u_R, x_i = 0 \forall i \in L, x_i = u_i \forall i \in U\}$
- So $d = b - a^T_U x_U$

Lifting

- Let $\pi^T x_R - \sigma^T y \leq \pi_0$ be a valid inequality for $K(L, U)$.
- Consider the lifting function $\Phi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$
  - $(\infty)$ if lifting problem is infeasible

$\Phi(\alpha) = \pi_0 - \max\{\pi^T_R x_R + \sigma^T y \mid a^T_R x_R + g^T y \leq d - \alpha, x_R \leq u_R, x_i \in \mathbb{Z}_+^{|R|}, y \in \mathbb{R}_+^{|M|}\}$
- In words, $\Phi(\alpha)$ is the maximum value of the LHS of the valid inequality if the RHS in $K$ is reduced by $\alpha$.

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- Why do we care about $\Phi$?
  
  \[
  \pi^T_R x_R + \pi^T_L x_L + \pi^T_U(u_U - x_U) + \sigma^T y \leq \pi_0
  \]
  
  is a valid inequality for $K$ if and only if
  
  \[
  \pi^T_L x_L + \pi^T_U(u_U - x_U) \leq \Phi(a^T_L x_L + a^T_U(x_U - u_U)) \forall (x, y) \in K.
  \]

Proof.?}

Example—Sequential Lifting

- Lifting one variable (at a time) in 0-1 IP (like we have done so far)...
  - $\alpha x_k + \pi^T_R x_R \leq \pi_0$ is valid for $P \iff \alpha x_k \leq \Phi(a_k x_k) \forall x \in P$
    - $x_k = 0, \quad 0 \leq \Phi(0)$ is always true.
    - $x_k = 1, \quad \Rightarrow \alpha \leq \Phi(a_i)$
  - If I “know” $\Phi(q)(\forall q \in \mathbb{R})$, I can just “lookup” the value of the lifting coefficient for variable $x_k$

Note that if I have restricted more than one variable, then this “lookup” logic is not necessarily true

- For lifting two (0-1) variables, I would have to look at four possible values.
- In general, the lifting function changes with each new variable “lifted.”
Superadditivity

- A function \( \phi : \mathbb{R} \to \mathbb{R} \) is superadditive if
  \[ \phi(q_1) + \phi(q_2) \leq \phi(q_1 + q_2) \]
- Superadditive functions play a significant role in the theory of integer programming. (See N&W page 229). (We'll probably revisit them later).
- Superadditive Fact:
  \[ \sum_{j \in N} \phi(a_j)x_j \leq \sum_{j \in N} \phi(a_jx_j) \leq \phi \left( \sum_{j \in N} a_jx_j \right) . \]

“Multiple Lookup”—Superadditivity

- Suppose that \( \phi \) is a superadditive lower bound on \( \Phi \) that satisfies \( \pi_i = \phi(a_i) \forall i \in L \) and \( \pi_i = \phi(-a_i) \forall i \in U \)
  \[ \sum_{i \in L} \phi(a_i)x_i + \sum_{i \in U} \phi(-a_i)(u_i - x_i) \leq \phi(a_L^Tx_L + a_U^T(x_U - u_U)) \]
  \[ \leq \Phi(a_L^Tx_L + a_U^T(x_U - u_U)) \]
- So
  \[ \pi_L^Tx_L + \pi_L^Tx_L + \pi_U^T(u_U - x_U) + \sigma^Ty \leq \pi_0 \]
  is a valid inequality for \( K \).

The Main Result

- If \( \phi \) is a superadditive lower bound on \( \Phi \), any inequality of the form \( \pi_R^Tx_R - \sigma^Ty \leq \pi_0 \), which is valid for \( K(L, U) \), can be extended to the inequality
  \[ \pi_R^Tx_R + \sum_{j \in L} \phi(a_j)x_j + \sum_{j \in U} \phi(-a_j)(u_j - x_j) + \sigma^Ty \leq \pi_0 \]
  which is valid for \( K \).
- If \( \pi_i = \phi(a_i) \forall i \in L \) and \( \pi_i = \phi(-a_i) \forall i \in U \) and
  \[ \pi^Tx_R - \sigma^Ty = \pi_0 \]
  defines a \( k \)-dimensional face of \( K(L, U) \), then the lifted inequality defines a face of dimension at least \( k + |L| + |U| \).

This Is Sooooooooooo Cool

- What does this imply?
  - If the lifting function itself is superadditive, I can lift all of the variables in one pass (if I know the lifting function, of course).
  - Even if I don’t know the lifting function, if I can get a superadditive function that is a lower bound, then I can lift all the variables at once.
Example

- This treatment follows that of Atamtürk’s paper I handed out.
- Cover \( C \) with \( \lambda = a(C) - b > 0 \)
  - Write our knapsack cover inequality as
    \[
    \sum_{j \in C} \lambda x_j \leq \lambda(|C| - 1)
    \]
- Lifting function \( \Theta : \mathbb{R} \to \mathbb{R} \cup \{\infty\} \)
  \[
  \Theta(\alpha) = \lambda(|C| - 1) - \max\{\sum_{j \in C} \lambda x_j | \sum_{j \in C} a_j x_j \leq b - \alpha\}.
  \]

Example—Lifted Knapsack Covers

\[
P = \text{conv}\{x \in \mathbb{B}^10 | 35x_1 + 27x_2 + 23x_3 + 19x_4 + 15x_5 + 15x_6 + 12x_7 + 8x_8 + 6x_9 + 3x_{10} \leq 39}\}
\]
- \( C = \{4, 5, 6\}, \) so \( \lambda = 10 \)
  \[
  \Theta(\alpha) = 20 - \max\{10x_4 + 10x_5 + 10x_6 | 35x_1 + 27x_2 + 23x_3 + 19x_4 + 15x_5 + 15x_6 + 12x_7 + 8x_8 + 6x_9 + 3x_{10} \leq 39 - \alpha\}
  \]
- AMPL Example...

Superadditive?

- Is \( \Theta(\alpha) \) superadditive?
  - No! \( \alpha_1 = 10, \alpha_2 = 25 \)
  \[
  \phi(\alpha) = \begin{cases} 0 & \text{if } 0 \leq \alpha \leq 9 \\ 10 + \alpha - 19 & \text{if } 9 \leq \alpha \leq 19 \\ 10 & \text{if } 19 \leq \alpha \leq 24 \\ 20 + \alpha - 34 & \text{if } 24 \leq \alpha \leq 34 \\ 20 & \text{if } 34 \leq \alpha \leq 39 \\ 30 + \alpha - 49 & \text{if } \alpha \geq 39 \end{cases}
  \]
  - Using \( \phi \) we get an inequality
    \[
    2x_1 + \frac{13}{10}x_2 + x_3 + x_4 + x_5 + x_6 + \frac{3}{10}x_7 \leq 2
    \]
How Strong?

- We know that the cover inequality $x_3 + x_4 + x_5 \leq 2$ defines a facet of the restricted problem.
- Is our inequality a facet of $K_2$?
  - Does $\phi(a_i) = \Phi(a_i) \ \forall i \in L$ and $\phi(-a_i) = \Phi(-a_i) \ \forall i \in U$?
- No! “Closest” facet is

$$2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 2$$