General Lifting and SuperAdditivity

- Let $\pi^T x_R - \sigma^T y \leq \pi_0$ be a valid inequality for $K(L, U)$.
- Consider the lifting function $\Phi : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$

$$\Phi(\alpha) = \pi_0 - \max\{\pi_R^T x_R + \sigma^T y \mid a_R^T x_R + g^T y \leq d - \alpha, x_R \leq u_R, x_R \in \mathbb{Z}_+^{|R|}, y \in \mathbb{R}_+^{|M|}\}$$

- $\infty$ if lifting problem is infeasible
- In words, $\Phi(\alpha)$ is the maximum value of the LHS of the valid inequality if the RHS in $K$ is reduced by $\alpha$.

Lifting

- $K = \text{conv}\{x \in \mathbb{Z}_+^{|N|}, y \in \mathbb{R}_+^{|M|} \mid a^T x + g^T y \leq b, x \leq u\}$
- Partition $N$ into $[L, U, R]$
  - $L = \{i \in N \mid x_i = 0\}$
  - $U = \{i \in N \mid x_i = u_i\}$
  - $R = N \setminus L \setminus U$
- We will use the notation: $x_R$ to mean the vector of variables that are in the set $R$.
  - $a^T_R x_R = \sum_{j \in R} a_j x_j$
  - $K(L, U) = \text{conv}\{x \in \mathbb{Z}_+^{|N|}, y \in \mathbb{R}_+^{|M|} \mid a^T_R x_R + g^T y \leq d, x_R \leq u_R, x_i = 0 \forall i \in L, x_i = u_i \forall i \in U.\}$

- So $d = b - a^T_U x_U$

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- Why do we care about $\Phi$?
  $$\pi^T_R x_R + \pi^T_L x_L + \pi^T_U (u_U - x_U) + \sigma^T y \leq \pi_0$$
is a valid inequality for $K$ if and only if
  $$\pi^T_L x_L + \pi^T_U (u_U - x_U) \leq \Phi(a^T_L x_L + a^T_U (x_U - u_U)) \forall (x, y) \in K.$$

Proof.

$$\Phi(a^T_L x_L + a^T_U (x_U - u_U)) = \pi_0 - \max\{\pi^T_R x_R + \sigma^T y \mid a^T_R x_R + g^T y \leq b - a^T_L x_L - a^T_U x_U, x_U \leq u_R, x_R \in \mathbb{Z}_+^{|R|}, y \in \mathbb{R}_+^{|M|}\}$$

So if there exists $(\hat{x}, \hat{y})$ such that
  $$\pi^T_L \hat{x}_L + \pi^T_U (u_U - \hat{x}_U) + \max\{\} > \pi_0,$$
then
  $$\pi^T_L \hat{x}_L + \pi^T_U (u_U - \hat{x}_U) + \pi^T_R x_R + \sigma^T y \leq \pi_0$$
cannot be a valid inequality.
Sequential Lifting. Example

- Suppose that we are doing sequential lifting for 0–1 IP like we have done so far.
- If $x_k$ fixed at 0. (Lower bound). $\alpha x_k + \pi^T_R x_R \leq \pi_0$ is valid for $P \iff \alpha x_k \leq \phi(a_k x_k) \forall x \in P$
  
  $x_k = 0$, 0 $\leq \phi(0)$ is always true.
  
  $x_k = 1$, $\Rightarrow \alpha \leq \phi(a_k)$

- If $x_k$ fixed at one (Upper Bound), then $\alpha(1 - x_k) + \pi^T_R x_R \leq \pi_0$ is valid for $P \iff \alpha(1 - x_k) \leq \phi(a_k(x_k - 1)) \forall x \in P$
  
  $x_k = 1$, 0 $\leq \phi(0)$ is always true.
  
  $x_k = 0$, $\Rightarrow \alpha \leq \phi(-a_k)$

- For some classes of inequalities, we have closed form solution for the lifting function.
- If I “know” $\phi(q)$ ($\forall q \in \mathbb{R}$), I can just “lookup” the value of the lifting coefficient for variable $x_k$.

Lifting Functions (Sequential)

- Note that if I have restricted more than one variable, then this “lookup” logic is not necessarily true.
  - For lifting two (0-1) variables, I would have to look at four possible values.
- In general, the lifting function $\Phi$ for some valid inequality $\pi^T_R x_R + g^T y \leq \pi_0$ changes as I lift variables: $\Phi_{i+1}(\alpha) \neq \Phi_i(\alpha) \forall i, \alpha$
- This implies that if I lift the variables in different orders, I can get different facets.

- What do we know about relationships between lifting functions?
  - It is monotonically decreasing: $\Phi_{i+1}(\alpha) \leq \Phi_i(\alpha) \forall i, \alpha$.
  - (Why?—N&W II.2, Proposition 1.3)
- The highest value a coefficient can have when I lift it comes when I lift it first.

Lifting Functions

- Suppose the lifting function doesn’t change when I lift a variable.
- If this happens, I can use the same lifting function again to determine the next coefficient.
- If the lifting function never changes, then I can use the same function to lift all of the variables.
- This happens if and only if $\Phi$ is a superadditive.

Superadditivity

- A function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is superadditive if
  
  $\phi(q_1) + \phi(q_2) \leq \phi(q_1 + q_2)$

- Superadditive functions play a significant role in the theory of integer programming. (See N&W page 229).
- Example: $\lfloor \cdot \rfloor$ is a superadditive function.
- Superadditive Fact:
  
  $\sum_{j \in N} \phi(a_j x_j) \leq \sum_{j \in N} \phi(a_j x_j) \leq \phi \left( \sum_{j \in N} a_j x_j \right)$. 

Note that if I have restricted more than one variable, then this “lookup” logic is not necessarily true.
“Multiple Lookup”—Superadditivity

- Suppose that $\phi$ is a superadditive lower bound on $\Phi$ that satisfies $\pi_i = \phi(a_i) \forall i \in L$ and $\pi_i = \phi(-a_i) \forall i \in U$

$$\sum_{i \in L} \phi(a_i)x_i + \sum_{i \in U} \phi(-a_i)(u_i - x_i) \leq \phi(a_L^Tx_L + a_U^T(x_U - u_U))$$

- So

$$\pi_R^Tx_R + \pi_L^Tx_L + \pi_U^T(u_U - x_U) + \sigma^Ty \leq \pi_0$$

is a valid inequality for $K$

The Main Result

- If $\phi$ is a superadditive lower bound on $\Phi$, any inequality of the form $\pi^Tx - \sigma^T y \leq \pi_0$, which is valid for $K(L,U)$, can be extended to the inequality

$$\pi_R^Tx_R + \sum_{j \in L} \phi(a_j)x_j + \sum_{j \in U} \phi(-a_j)(u_j - x_j) + \sigma^Ty \leq \pi_0$$

which is valid for $K$.

- If $\phi(a_i) = \Phi(a_i) \forall i \in L$ and $\phi(-a_i) = \Phi(a_i) \forall i \in U$ and $\pi^Tx - \sigma^Ty = \pi_0$ defines a $k$-dimensional face of $K(L, U)$, then the lifted inequality defines a face of dimension at least $k + |L| + |U|$ of $K$.

Example—Lifted Knapsack Covers

- What does this imply?

  - If the lifting function itself is superadditive, I can lift all of the variables in one pass (if I know the lifting function, of course).
  - Even if I don’t know the lifting function, if I can get a superadditive function that is a lower bound, then I can lift all the variables at once.
  - Often, by examining the special structure of the lifting problem, one can fairly easily deduce a (closed form) solution for the lifting function.
  - Then one can also deduce a superadditive lower bound

$$P = \text{conv}\{x \in \mathbb{B}^{10} \mid 35x_1 + 27x_2 + 23x_3 + 19x_4 + 15x_5 + 15x_6 + 12x_7 + 8x_8 + 6x_9 + 3x_{10} \leq 39\}$$

- $C = \{4, 5, 6\}$, so $\lambda = 10$

$$\Theta(\alpha) = 20 - \max\{10x_4 + 10x_5 + 10x_6 \mid 35x_1 + 27x_2 + 23x_3 + 19x_4 + 15x_5 + 15x_6 + 12x_7 + 8x_8 + 6x_9 + 3x_{10} \leq 39 - \alpha\}$$
Superadditive?

- Is $\Theta(\alpha)$ superadditive?
  - No! $\alpha_1 = 10, \alpha_2 = 25$

$$\phi(\alpha) = \begin{cases} 
0 & \text{if } 0 \leq \alpha \leq 9 \\
10 + \alpha - 19 & \text{if } 9 \leq \alpha \leq 19 \\
10 & \text{if } 19 \leq \alpha \leq 24 \\
20 + \alpha - 34 & \text{if } 24 \leq \alpha \leq 34 \\
20 & \text{if } 34 \leq \alpha \leq 39 \\
30 + \alpha - 49 & \text{if } \alpha \geq 39 
\end{cases}$$

- Using $\phi$ we get an inequality
  $$2x_1 + \frac{13}{10}x_2 + x_3 + x_4 + x_5 + x_6 + \frac{3}{10}x_7 \leq 2$$

Facets of $P$

- We often try to solve problems that have knapsack rows with lots more variables than that...
- Obviously I do not want to add all of those facets.
- What to do?
  - Given some $\hat{x} \notin P$, find an inequality of the form
    $$\sum_{j \in C} x_j \leq |C| - 1$$
  - This is called a separation problem
  - Note that it is dependent on the particular class of inequalities—in this case cover inequalities.
Knapsack Separation

- Note that $\sum_{j \in C} x_j \leq |C| - 1$ can be rewritten as
  $$\sum_{j \in C} (1 - x_j) \geq 1.$$

- Separation Problem: Given a “fractional” LP solution $\hat{x}$, does $\exists C \subseteq N$ such that $\sum_{j \in C} a_j > b$ and $\sum_{j \in C} (1 - \hat{x}_j) < 1$?

- Is $\gamma = \min_{C \subseteq N} \{\sum_{j \in C} (1 - \hat{x}_j) \mid \sum_{j \in C} a_j > b\} < 1$?

- Let $z_j \in \{0, 1\}$, $z_j = 1$ if $j \in C$, $z_j = 0$ if $j \notin C$.

- Is $\gamma = \min \{\sum_{j \in N} (1 - \hat{x}_j)z_j \mid \sum_{j \in N} a_jz_j > b, z \in \mathbb{B}^n\} < 1$?

- If $\gamma \geq 1$, $\hat{x}$ satisfies all cover inequalities

- If $\gamma < 1$ with optimal solution $z_R$, then $\sum_{j \in R} x_j \leq |R| - 1$ is a violated cover inequality.

Complexity of Separation

- How hard is it to separate a fractional LP solution?
- Is it obvious that it is hard?
  - **No!** Since the point you are trying to separate is not an “arbitrary” knapsack problem, but instead the profits have a special form.
  - Klabjan, Nemhauser, and Tovey, “The Complexity of Cover Inequality Separation” show that knapsack separation is NP-Hard.

Example

Example: $\text{MYKNAP} = \{x \in \mathbb{B}^7 \mid 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\}$

- $\hat{x} = (0, 2/3, 0, 1, 1, 1, 1)$

- $\gamma = \min_{z \in \mathbb{B}^7} \{z_1 + 1/3z_2 + z_3 \mid 11z_1 + 6z_2 + 6z_3 + 5z_4 + 5z_5 + 4z_6 + z_7 \geq 20\}.$

- $\gamma = 1/3$

- $z = (0, 1, 0, 1, 1, 1, 1)$

- $x_2 + x_4 + x_5 + x_6 + x_7 \leq 4$

- Minimal Cover: $x_2 + x_4 + x_5 + x_6 \leq 3$

- You would do the lifting from here.