Stochastic Programming Modeling

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Outline

- Review convexity
- Review Farmer Ted
- *Expected Value of Perfect Information*
- *Value of the Stochastic Solution*
- Building the Deterministic Equivalent
  - In an algebraic modeling language
- Formal notation
- More examples
• Name one way in which to deal with randomness in mathematical programming problems.

• Name another way.

• Name another way

• A set $C$ is convex if and only if...

• A function $f$ is convex if and only if...

• What does Farmer Ted like to grow?
For the Math Lovers Out There...

- It is *extremely* important to understand the convexity properties of a function you are trying to optimize.

- A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is *convex* if for any two points \( x \) and \( y \), the graph of \( f \) lies below or on the straight line connecting \((x, f(x))\) to \((y, f(y))\) in \( \mathbb{R}^{n+1} \).
  
  \[ f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) \quad \forall 0 \leq \alpha \leq 1 \]

- A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is *concave* if for any two points \( x \) and \( y \), the graph of \( f \) lies above or on the straight line connecting \((x, f(x))\) to \((y, f(y))\) in \( \mathbb{R}^{n+1} \).
  
  \[ f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha)f(y) \quad \forall 0 \leq \alpha \leq 1 \]

- A function that is neither convex nor concave, we will call *nonconvex*.
• A set $S$ is *convex* if the straight line segment connecting any two points in $S$ lies entirely inside or on the boundary of $S$.
  \[ x, y \in S \Rightarrow \alpha x + (1 - \alpha)y \in S \quad \forall 0 \leq \alpha \leq 1 \]

• A Confusing Point...
  ◦ Why do they have a *convex function* and a *convex set*? How are they related?
  ◦ $f$ is convex if and only if the *epigraph*, or “over part” of $f$ is a convex set.
CONVEX

NONCONVEX
True or False

- Discrete Constraint Sets are convex?
- Empty Constraint Sets are convex?
- Discontinuous functions are convex?
Recall Farmer Ted

- Farmer Ted can grow Wheat, Corn, or Beans on his 500 acres.
- Farmer Ted requires 200 tons of wheat and 240 tons of corn to feed his cattle
  - These can be grown on his land or bought from a wholesaler.
  - Any production in excess of these amounts can be sold for $170/ton (wheat) and $150/ton (corn)
  - Any shortfall must be bought from the wholesaler at a cost of $238/ton (wheat) and $210/ton (corn).
- Farmer Ted can also grow beans
  - Beans sell at $36/ton for the first 6000 tons
  - Due to economic quotas on bean production, beans in excess of 6000 tons can only be sold at $10/ton
Formulate the LP – Decision Variables

- $x_{W,C,B}$ Acres of Wheat, Corn, Beans Planted
- $w_{W,C,B}$ Tons of Wheat, Corn, Beans sold (at favorable price).
- $e_B$ Tons of beans sold at lower price
- $y_{W,C}$ Tons of Wheat, Corn purchased.
maximize
\[-150x_W - 230x_C - 260x_B - 238y_W + 170w_W - 210y_C + 150y_C + 36w_B + 10e_B\]
subject to
\[
\begin{align*}
x_W + x_C + x_B & \leq 500 \\
2.5x_W + y_W - w_W & = 200 \\
3x_C + y_C - w_C & = 240 \\
20x_B - w_B - e_B & = 0 \\
w_B & \leq 6000 \\
x_W, x_C, x_B, y_W, y_C, e_B, w_W, w_C, w_B & \geq 0
\end{align*}
\]
Randomness

- Farmer Ted knows he doesn’t get the yields $Y$ all the time.
- Assume that three yield scenarios $(1.2Y, Y, 0.8Y)$ occur with equal probability.
- Maximize $Expected$ Profit
- Attach a scenario subscript $s = 1, 2, 3$ to each of the purchase and sale variables.
  - ◊ 1: Good, 2: Average, 3: Bad
  
  Ex. $w_{C2}$ : Tons of corn sold at favorable price in scenario 2
  
  Ex. $e_{B3}$ : Tons of beans sold at unfavorable price in scenario 3.
• An expression for Farmer Ted’s Expected Profit is the following:

\[-150x_W - 230x_C - 260x_B
+ \frac{1}{3}(-238y_{W1} + 170w_{W1} - 210y_{C1} + 150y_{C1} + 36w_{B1} + 10e_{B1})
+ \frac{1}{3}(-238y_{W2} + 170w_{W2} - 210y_{C2} + 150y_{C2} + 36w_{B2} + 10e_{B2})
+ \frac{1}{3}(-238y_{W3} + 170w_{W3} - 210y_{C3} + 150y_{C3} + 36w_{B3} + 10e_{B3})\]
Expected Value Problem – Constraints

\[ x_W + x_C + x_B \leq 500 \]
\[ 3x_W + y_{W1} - w_{W1} = 200 \]
\[ 2.5x_W + y_{W2} - w_{W2} = 200 \]
\[ 2x_W + y_{W3} - w_{W3} = 200 \]
\[ 3.6x_C + y_{C1} - w_{C1} = 240 \]
\[ 3x_C + y_{C2} - w_{C2} = 240 \]
\[ 2.4x_C + y_{C3} - w_{C3} = 240 \]
\[ 24x_B - w_{B1} - e_{B1} = 0 \]
\[ 20x_B - w_{B2} - e_{B2} = 0 \]
\[ 16x_B - w_{B3} - e_{B3} = 0 \]
\[ w_{B1}, w_{B2}, w_{B3} \leq 6000 \]
\[ \text{All vars} \geq 0 \]
### Optimal Solution

<table>
<thead>
<tr>
<th></th>
<th>Wheat</th>
<th>Corn</th>
<th>Beans</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>Plant (acres)</td>
<td>100</td>
<td>25</td>
</tr>
<tr>
<td>1</td>
<td>Production</td>
<td>510</td>
<td>288</td>
</tr>
<tr>
<td>1</td>
<td>Sales</td>
<td>310</td>
<td>48</td>
</tr>
<tr>
<td>1</td>
<td>Purchase</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>Production</td>
<td>425</td>
<td>240</td>
</tr>
<tr>
<td>2</td>
<td>Sales</td>
<td>225</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>Purchase</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>Production</td>
<td>340</td>
<td>192</td>
</tr>
<tr>
<td>3</td>
<td>Sales</td>
<td>140</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>Purchase</td>
<td>0</td>
<td>48</td>
</tr>
</tbody>
</table>

- (Expected) Profit: $108390
• Congratulations, we’ve just solved our first stochastic program.

• What we’ve done is known as forming (and solving) the deterministic equivalent of a stochastic program.

• Note that you can always do this when...
  ◦ $\Omega$ is a finite set. (There are a finite number of scenarios $\omega_1, \omega_2, \ldots, \omega_K \in \Omega$)
  ◦ We are interested in optimizing an expected value.

  $\Rightarrow$ We can write $\mathbb{E}_\omega f(x, \omega)$ as $\sum_{k=1}^{K} p_k f(x, \omega_k)$
Wait and See

- Recall from last time, that Farmer Ted also “ran some scenarios”

- *Given that he knew the yields*, what was his best policy?
  - We called these “Wait-and-see” solutions

<table>
<thead>
<tr>
<th></th>
<th>0.8Y</th>
<th>Y</th>
<th>1.2Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corn</td>
<td>25</td>
<td>80</td>
<td>66.67</td>
</tr>
<tr>
<td>Wheat</td>
<td>100</td>
<td>120</td>
<td>183.33</td>
</tr>
<tr>
<td>Beans</td>
<td>375</td>
<td>300</td>
<td>250</td>
</tr>
<tr>
<td>Profit</td>
<td>59950</td>
<td>118600</td>
<td>167667</td>
</tr>
</tbody>
</table>
Suppose Farmer Ted could *with certainty* tell whether or not the upcoming growing season was going to be wet, average, or dry (or what his yields were going to be).

- His bursitis was acting up
- Consulting the Farmer’s Almanac
- Hiring a fortune teller

The real point here is how *much* Farmer Ted would be willing to pay for this “perfect” information.

In real-life problems, how much is it “worth” to invest in better forecasting technology?

This amount is called *The Expected Value of Perfect Information*. 
What is the EVPI?

• With perfect information, Farmer Ted’s Long Run Profit/Year would be:
  \[ \frac{1}{3}(167667) + \frac{1}{3}(118600) + \frac{1}{3}(59950) = 115406 \]

• Without perfect information, Farmer Ted can at best maximize his expected profit by solving the stochastic program.

• In this case, he would make 108390 in the long run
  \[ \text{EVPI} = 115406 - 108390 = 7016. \]

• Is there any other important information that you would like to know?
  \[ \text{What is the value of including the randomness?} \]
The Value of the Stochastic Solution (VSS)

- Suppose we just replaced the “random” quantities (the yields) by their mean values and solved that problem.
- Would we get the same expected value for the Farmer’s profit?
- How can we check?
  - Solve the “mean-value” problem to get a first stage solution $x$. (A “policy”).
  - Fix the first stage solution at that value $x$, and solve all the scenarios to see Farmer Ted’s profit in each.
  - Take the weighted (by probability) average of the optimal objective value for each scenario.
To do this, we’ll use AMPL

- You are welcome to solve problems anyway you can
  - Except for copying/cheating
  - An algebraic modeling language will be quite useful!
- Average AMPL proficiency was around 7, and minimum was 3, so I am going to assume everyone comfortable with AMPL.
- There are some AMPL pointers on the web page.
- I have one copy of the AMPL book I can loan out for brief periods.
- AMPL is all about algebraic notation, so let’s convert Farmer Ted to a more algebraic description...
• Sets...
  ◇ $C$: Set of crops
  ◇ $D \subseteq C$: Set of crops that have quotas
  ◇ $Q \subseteq C$: Set of crops that FT can purchase.

• Variables...
  ◇ $x_c, c \in C$: Acres to allocate to $c$
  ◇ $w_c, c \in C$: Amount of $c$ to sell (at high price)
  ◇ $y_c, c \in C: (y_c = 0 \ \forall \ c \in C \setminus Q)$: Amount of $c$ to purchase
  ◇ $e_c, c \in C: (e_c = 0 \ \forall \ c \in D)$: Amount of $c$ to sell (at low price)
AMPL

(Showing off AMPL here)
Great, but This Class is called *Stochastic* Programming

- Here’s how to create the deterministic equivalent...

- For each possible state of nature (scenario), formulate an appropriate LP model

- Combine these submodels into one “supermodel” making sure
  - The first-stage variables are common to all submodels
  - The second-stage variables in a submodel appear only in that submodel

  - Do this by attaching a “scenario index” to the second stage variables and to the parameters that change in the different scenarios
Deterministic Equivalent

- Combine these submodels into one “supermodel” making sure:
  - The first-stage variables are common to all submodels
  - The second-stage variables in a submodel appear only in that submodel

\[
\begin{align*}
x_W + x_C + x_B & \leq 500 \\
3x_W + y_{W1} - w_{W1} & = 200 \\
2.5x_W + y_{W2} - w_{W2} & = 200 \\
2x_W + y_{W3} - w_{W3} & = 200
\end{align*}
\]
3.6x_C + y_{C1} - w_{C1} = 240
3x_C + y_{C2} - w_{C2} = 240
2.4x_C + y_{C3} - w_{C3} = 240
24x_B - w_{B1} - e_{B1} = 0
20x_B - w_{B2} - e_{B2} = 0
16x_B - w_{B3} - e_{B3} = 0
w_{B1}, w_{B2}, w_{B3} \leq 6000
All vars \geq 0
Computing Farmer Ted’s VSS

- Solve the “mean-value” problem to get a first stage solution $x$. (A “policy”).
  - Mean yields $Y = (2.5, 3, 20)$
  - (We already solved this problem).
  - $x_W = 120$, $x_C = 80$, $x_B = 300$
- Fix the first stage solution at that value $x$, and solve all the scenarios to see Farmer Ted’s profit in each.
- Take the weighted (by probability) average of the optimal objective value for each scenario
Fixed Policy – Average Yield Scenario

maximize

\[-150x_W - 230x_C - 260x_B - 238y_W + 170w_W - 210y_C + 150y_C + 36w_B + 10e_B\]

subject to

\[
\begin{align*}
x_W &= 120 \\
x_C &= 80 \\
x_B &= 300 \\
x_W + x_C + x_B &\leq 500 \\
2.5x_W + y_W - w_W &= 200 \\
3x_C + y_C - w_C &= 240 \\
20x_B - w_B - e_B &= 0 \\
w_B &\leq 6000 \\
x_W, x_C, x_B, y_W, y_C, e_B, w_W, w_C, w_B &\geq 0
\end{align*}
\]
Fixed Policy – Bad Yield Scenario

maximize

\[-150x_W - 230x_C - 260x_B - 238y_W + 170w_W - 210y_C + 150y_C + 36w_B + 10e_B\]

subject to

\[
\begin{align*}
x_W &= 120 \\
x_C &= 80 \\
x_B &= 300 \\
x_W + x_C + x_B &\leq 500 \\
2x_W + y_W - w_W &= 200 \\
2.4x_C + y_C - w_C &= 240 \\
16x_B - w_B - e_B &= 0 \\
w_B &\leq 6000 \\
x_W, x_C, x_B, y_W, y_C, e_B, w_W, w_C, w_B &\geq 0
\end{align*}
\]
Fixed Policy – Good Yield Scenario

maximize

\[-150x_W - 230x_C - 260x_B - 238y_W + 170w_W - 210y_C + 150y_C + 36w_B + 10e_B\]

subject to

\[
\begin{align*}
x_W &= 120 \\
x_C &= 80 \\
x_B &= 300 \\
x_W + x_C + x_B &\leq 500 \\
3x_W + y_W - w_W &= 200 \\
3.6x_C + y_C - w_C &= 240 \\
24x_B - w_B - e_B &= 0 \\
w_B &\leq 6000 \\
x_W, x_C, x_B, y_W, y_C, e_B, w_W, w_C, w_B &\geq 0
\end{align*}
\]
- If you solved those three problems, you would get

<table>
<thead>
<tr>
<th>Yield</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>118600</td>
</tr>
<tr>
<td>Bad</td>
<td>55120</td>
</tr>
<tr>
<td>Good</td>
<td>148000</td>
</tr>
</tbody>
</table>

- Another trick – you don’t need to solve all three. Just solve the DE with the first stage fixed.

- I’ll show you this if we have time.
What’s it Worth to Model Randomness?

- If Farmer Ted implemented the policy based on using only “average” yields, he would plant $x_W = 120, x_C = 80, x_B = 300$
- He would expect in the long run to make an average profit of...
  - $\frac{1}{3}(118600) + \frac{1}{3}(55120) + \frac{1}{3}(148000) = 107240$
- If Farmer Ted implemented the policy based on the solution to the stochastic programming problem, he would plant $x_W = 170, x_C = 80, x_B = 250$.
  - From this he would expect to make 108390
The difference of the values 180390-107240 is the Value of the Stochastic Solution: $1150.

- It would pay off $1150 per growing season for Farmer Ted to use the “stochastic” solution rather than the “mean value” solution.