A Characterization of Deterministic Sampling Patterns for Low-Rank Matrix Completion

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Abstract—Low-rank matrix completion (LRMC) problems arise in a wide variety of applications. Previous theory mainly provides conditions for completion under missing-at-random samplings. This paper studies deterministic conditions for completion. An incomplete \( d \times N \) matrix is \textit{finitely rank-} \( r \) \textit{completable} if there are at most finitely many rank-\( r \) matrices that agree with all its observed entries. Finite completability is the tipping point in LRMC, as a few additional samples of a finitely completable matrix guarantee its \textit{unique} completability. The main contribution of this paper is a characterization of finitely completable observation sets. We use this characterization to derive sufficient deterministic sampling conditions for unique completability. We also show that under uniform random sampling schemes, these conditions are satisfied with high probability if \( \mathcal{O}(\max\{r, \log d\}) \) entries per column are observed.

I. INTRODUCTION

Low-rank matrix completion (LRMC) has attracted a lot of attention in recent years because of its broad range of applications, e.g., recommender systems and collaborative filtering [1] and image processing [2].

The problem entails exactly recovering all the entries in a \( d \times N \) rank-\( r \) matrix, given only a subset of its entries. LRMC is usually studied under a missing-at-random and bounded-coherence model. Under this model, necessary and sufficient conditions for perfect recovery are known [3]–[8]. Other approaches require additional coherence and spectral gap conditions [9], use rigidity theory [10], algebraic geometry and matroid theory [11] to derive necessary and sufficient conditions for completion of deterministic samplings, but a characterization of completable sampling patterns remained an important open question until now.

We say an incomplete matrix is \textit{finitely rank-} \( r \) \textit{completable} if there exist at most finitely many rank-\( r \) matrices that agree with all its observed entries. Finite completability is the tipping point in LRMC. If even a single observation of a finitely completable matrix is instead missing, then there exist \textit{infinitely} many completions. Conversely, just a few additional samples of a finitely completable matrix guarantee its \textit{unique} completability.

Whether a matrix is finitely completable depends on which entries are observed. Yet characterizing the sets of observed entries that allow or prevent finite completability remained an important open question until now.

The main result of this paper is a characterization of finitely completable observation sets, that is, sampling patterns that can be completed in at most finitely many ways. In addition, we provide deterministic sampling conditions for unique completability. Finally, we show that uniform random samplings with \( \mathcal{O}(\max\{r, \log d\}) \) entries per column satisfy these conditions with high probability.

Organization of the Paper

In Section II we formally state the problem and our main results. We present the proof of our main theorem in Section III, and we leave the proofs of our other statements to Sections IV and V, where we also present an additional useful sufficient condition for finite completability.

II. MODEL AND MAIN RESULTS

Let \( \mathbf{X}_\Omega \) denote the incomplete version of a \( d \times N \) rank-\( r \) data matrix \( \mathbf{X} \), observed only in the nonzero locations of \( \Omega \). a \( d \times N \) matrix with binary entries. First observe that since \( \mathbf{X} \) is rank-\( r \), a column with fewer than \( r \) samples cannot be completed. We will thus assume without loss of generality that

A1 Every column of \( \mathbf{X} \) is observed in at least \( r \) entries.

The LRMC problem is tantamount to identifying the \( r \)-dimensional subspace \( S^r \) spanned by the columns in \( \mathbf{X} \), and this is how we will approach it. The key insight of the paper is that observing more than \( r \) entries in a column of \( \mathbf{X} \) places constraints on what \( S^r \) may be. For example, if we observe \( r+1 \) entries of a particular column, then not all \( r \)-dimensional subspaces will be consistent with the entries. If we observe more entries, then even fewer subspaces will be consistent with them. In effect, each observed entry, in addition to the first \( r \) observations, places one constraint that an \( r \)-dimensional subspace must satisfy in order to be consistent with the observations. The observed entries in different columns may or may not produce redundant constraints. The main result of this paper is a simple condition on the set of constraints (resulting from all the observations) that is necessary and sufficient to guarantee that only a finite number of subspaces satisfies all the constraints. This in turn provides a simple condition for exact matrix completion.

To state the result, we introduce the matrix \( \tilde{\Omega} \) that encodes the set of all constraints in a way that allows us to easily express the necessary and sufficient condition. Let \( k_1, \ldots, k_{\ell_i} \) denote the indices of the \( \ell_i \) observed entries in the \( i^{th} \) column of \( \mathbf{X} \). Define \( \Omega_i \) as the \( d \times (\ell_i - r) \) matrix, whose \( j^{th} \) column has the value 1 in rows \( k_1, \ldots, k_r \) and \( k_{r+j} \), and zeros elsewhere. For example, if \( k_1 = 1, k_2 = 2, \ldots, k_{\ell_i} = \ell_i \),
then

\[ \Omega_i = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} r \\ \ell_i - r \\ \ell_i - r \\ d - \ell_i \end{bmatrix}, \]

where 1 denotes a block of all 1’s and I the identity matrix. Finally, define \( \hat{\Omega} := [\Omega_1 \cdots \Omega_N] \). The matrix \( \hat{\Omega} \) encodes all the constraints placed on subspaces consistent with the observations, and as we will see, the pattern of nonzero entries determines whether or not the constraints are redundant, thus indicating the number of subspaces that satisfy them.

Let \( \text{Gr}(r, \mathbb{R}^d) \) denote the Grassmannian manifold of \( r \)-dimensional subspaces in \( \mathbb{R}^d \). Observe that each \( d \times N \) rank-\( r \) matrix \( \mathbf{X} \) can be uniquely represented in terms of a subspace \( S^* \in \text{Gr}(r, \mathbb{R}^d) \) (spanning the columns of \( \mathbf{X} \)) and an \( r \times N \) coefficient matrix \( \Theta^* \). Let \( \nu_G \) denote the uniform measure on \( \text{Gr}(r, \mathbb{R}^d) \), and let \( \nu_\Theta \) denote the Lebesgue measure on \( \mathbb{R}^{r \times N} \). Our statements hold for almost every (a.e.) \( \mathbf{X} \) with respect to the product measure \( \nu_G \times \nu_\Theta \).

The paper’s main result is the following theorem, which gives a deterministic necessary and sufficient sampling condition to guarantee that at most a finite number of \( r \)-dimensional subspaces are consistent with \( \mathbf{X}_\Omega \).

Given a matrix, let \( n(\cdot) \) denote its number of columns and \( m(\cdot) \) the number of its nonzero rows.

**Theorem 1.** Let \( \Omega \) be given, and suppose \( A1 \) holds. For almost every \( \mathbf{X} \), there exist at most finitely many rank-\( r \) completions of \( \mathbf{X}_\Omega \) if and only if there is a matrix \( \hat{\mathbf{X}} \) formed with \( r(d-r) \) columns of \( \hat{\Omega} \), such that every matrix \( \Omega' \) formed with a subset of the columns in \( \hat{\Omega} \) satisfies

\[ m(\Omega') \geq n(\Omega')/r + r. \] (1)

The proof of Theorem 1 is given in Section III. The condition on \( \hat{\Omega} \) in Theorem 1 is that every subset of \( n \) columns of \( \hat{\Omega} \) must have at least \( n/r + r \) nonzero rows.

**Example 1.** The following sampling satisfies the conditions of Theorem 1, where \( \hat{\Omega} = \hat{\Omega} = \Omega \).

\[ \Omega = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ r & r & \ldots & r \end{bmatrix} \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ d - r \end{bmatrix} \begin{bmatrix} \ell_1 - r & \ell_2 - r & \ldots & \ell_N - r \end{bmatrix}. \]

**Unique Completability**

Theorem 1 is easily extended to a condition on \( \hat{\Omega} \) that is sufficient to guarantee that one and only one subspace is consistent with \( \mathbf{X}_\Omega \), which in turn suffices for exact matrix completion.

**Theorem 2.** Let \( \Omega \) be given, and suppose \( A1 \) holds. Then almost every \( \mathbf{X} \) can be uniquely recovered from \( \mathbf{X}_\Omega \) if \( \hat{\Omega} \) contains two disjoint submatrices: \( \hat{\Omega} \) of size \( d \times r(d-r) \) and \( \hat{\Omega} \) of size \( d \times (d-r) \), such that the following two conditions are satisfied.

(i) Every matrix \( \Omega' \) formed with a subset of the columns in \( \hat{\Omega} \) satisfies

\[ m(\Omega') \geq n(\Omega') + r. \] (2)

(ii) Every matrix \( \Omega' \) formed with a subset of the columns in \( \hat{\Omega} \) satisfies

\[ m(\Omega') \geq n(\Omega') + r. \] (1)

The proof of Theorem 2 is given in Section IV. Condition (ii) in Theorem 2 is that every subset of \( n \) columns of \( \hat{\Omega} \) must have at least \( n + r \) nonzero rows. Notice that (1) is a weaker condition than (2), but (1) is required to hold for all the subsets of \( r(d-r) \) columns, while (2) is required to hold only for all the subsets of \( d-r \) columns.

**Example 2.** A sampling with the same pattern as in Example 1, but with \( (r+1)(d-r) \) columns, satisfies the conditions of Theorem 2.

Defining \( \mathcal{N} \) as the number of columns in \( \hat{\Omega} \), Theorem 1 implies that \( \mathcal{N} = r(d-r) \) columns are necessary for finite completableility (hence also for unique completability). There are cases when \( \mathcal{N} = r(d-r) \) is also sufficient for unique completability, e.g., if \( r = 1 \), where finite completableility is equivalent to unique completability (see Proposition 1).

In general, though, \( \mathcal{N} > r(d-r) \) columns are necessary for unique completability (see Example 3). Theorem 2 gives deterministic sampling sufficient conditions for unique completableility that only require \( \mathcal{N} = (r+1)(d-r) \) columns. This shows that with just a few more observations, unique completability follows from finite completableility.

Furthermore, when the conditions of Theorem 2 are met, \( S^* \) can be uniquely identified (and thus \( \mathbf{X} \) can be uniquely recovered) as

\[ S^* = \text{span} \begin{bmatrix} 1 \\ V \end{bmatrix}, \]

where \( V \) is the unique solution to the polynomial system \( \mathcal{F}(V) = 0 \), with \( \mathcal{F} \) as defined in Section III.

**Random Sampling Patterns**

In general, verifying the conditions in Theorems 1 and 2 may be computationally prohibitive, especially for large \( d \). However, as the next theorem states, sampling patterns satisfying these conditions appear with high probability under uniform random sampling schemes with only \( O(\max\{r, \log d\}) \) samples per column.
Theorem 3. Let $0 < \epsilon \leq 1$ be given. Suppose $r \leq \frac{d}{\epsilon}$ and that each column of $X$ is observed in at least $\ell$ entries, distributed uniformly at random and independently across columns, with

$$\ell \geq \max\left\{ 12 \log\left( \frac{2}{\epsilon} \right) + 1, 2r \right\}.\tag{3}$$

Then with probability at least $1 - \epsilon$, $X|_{\Omega}$ will be finitely rank-$r$ completable (if $N \geq r(d - r)$) and uniquely completable (if $N \geq (r + 1)(d - r)$).

Theorem 3 is proved in Section V.

III. PROOF OF THEOREM 1

For any subspace, matrix or vector that is compatible with a set of indices $\omega$, we will use the subscript $\omega$ to denote its restriction to the coordinates/rows in $\omega$. For example, letting $\omega_i$ denote the indices of the nonzero rows of the $i^{\text{th}}$ column of $\Omega$, then $x_{\omega_i} \in \mathbb{R}^\ell_i$ and $S_{\omega_i}^\ell \in \mathbb{R}^{\ell_i}$ denote the restrictions of the $i^{\text{th}}$ column in $X$ and $S^*$, to the indices in $\omega_i$. We say that an $r$-dimensional subspace $S$ fits $X|_{\Omega}$ if $x_{\omega_i} \in S_{\omega_i}$, $\forall i$.

The Variety $S$

Let us start by studying the variety of all $r$-dimensional subspaces that fit $X|_{\Omega}$. First observe that in general, the restriction of an $r$-dimensional subspace to $\ell \leq r$ coordinates is $\mathbb{R}^\ell$. We formalize this in the following definition, which essentially states that a subspace is non-degenerate if its restrictions to $\ell \leq r$ coordinates are $\mathbb{R}^\ell$.

Definition 1 (Degenerate subspace). We say $S \subseteq \text{Gr}_r(r, \mathbb{R}^d)$ is degenerate if and only if there exists a set $\omega \subseteq \{1, \ldots, d\}$ with $|\omega| \leq r$, such that $\dim S_{\omega} \leq |\omega|$.

Let $\nu_{S}$ denote the uniform measure on $\text{Gr}_r(r, \mathbb{R}^d)$. A subspace is degenerate if and only if an $r \times r$ submatrix of one of its bases is rank-deficient. This is equates to having a zero determinant. Since the determinant is a polynomial in the entries of a matrix, this is a condition of $\nu_S$-measure zero.

Since $\nu_{S}$-almost every subspace is non-degenerate, let us consider only the subspaces in $\text{Gr}_r(r, \mathbb{R}^d) \subseteq \text{Gr}_r(r, \mathbb{R}^d)$: the set of all non-degenerate $r$-dimensional subspaces of $\mathbb{R}^d$.

Define $S(X|_{\Omega}) \subseteq \text{Gr}_r(r, \mathbb{R}^d)$ such that every $S \in S(X|_{\Omega})$ fits $X|_{\Omega}$, i.e.,

$$S(X|_{\Omega}) : = \left\{ S \in \text{Gr}_r(r, \mathbb{R}^d) : \{ x_{\omega_i} \in S_{\omega_i} : i = 1 \} \right\}.$$

Let $U \in \mathbb{R}^{d \times x}$ be a basis of $S \in S(X|_{\Omega})$. The condition $x_{\omega_i} \in S_{\omega_i}$ is equivalent to saying that there exists a vector $\theta_i \in \mathbb{R}^r$ such that

$$x_{\omega_i} = U|_{\omega_i}\theta_i.\tag{4}$$

We can see that if $x_{\omega_i}$ has fewer than $r$ observations, (4) will be an underdetermined system with infinitely many solutions, and hence $x_{\omega_i}$ can be completed in infinitely many ways.

If $x_{\omega_i}$ has exactly $r$ observations, (4) becomes a system with $r$ equations and $r$ unknowns (the elements of $\theta_i$). This will be the case for every $S \in \text{Gr}_r(r, \mathbb{R}^d)$. Hence a column with exactly $r$ observations can be uniquely completed once $S^*$ is known, but it provides no information to identify $S^*$.

On the other hand, if $x_{\omega_i}$ has exactly $r + 1$ observations, then (4) becomes an overdetermined system with $r + 1$ equations and $r$ unknowns. This imposes one constraint on the elements of $U|_{\omega_i}$, thus restricting the set of subspaces that fit $x_{\omega_i}$.

In general, a column with $\ell_i \geq r$ observations will impose $\ell_i - r$ constraints, each of which may reduce one out of the $r(d - r)$ degrees of freedom in $\text{Gr}_r(r, \mathbb{R}^d)$. Therefore, one necessary condition for completion is that $X|_{\Omega}$ imposes at least $r(d - r)$ constraints, i.e., that

$$\sum_{i=1}^N (\ell_i - r) \geq r(d - r).$$

We will now study these constraints and characterize when exactly will they reduce all the $r(d - r)$ degrees of freedom in $\text{Gr}_r(r, \mathbb{R}^d)$, thus restricting $S(X|_{\Omega})$ to a set with at most finitely many elements.

Let $\Delta_i$ be a set with $r$ elements of $\omega_i$, and let $\{ \omega_i^{(1)}, \ldots, \omega_i^{(r)} \}$ denote a partition of the remaining elements of $\omega_i$. We can then expand (4) as

$$r \left[ \begin{array}{c} x_{\Delta_i} \\ 1 \\ \vdots \\ 1 \end{array} \right] = \left[ \begin{array}{c} U_{\Delta_i} \\ U_{\nu_i} \\ \vdots \\ U_{\nu_i(\ell_i-1)} \end{array} \right] \theta_i.$$

Since $S$ is non-degenerate, $U_{\Delta_i}$ is full-rank, so we may solve for $\theta_i$ using the top block to obtain $\theta_i = U_{\Delta_i}^{-1}x_{\Delta_i}$. Plugging this on the remaining rows, we have that (4) is equivalent to:

$$\{ x_{\nu_i} = U_{\nu_i}U_{\Delta_i}^{-1}x_{\Delta_i} \}_{j=1}^{\ell_i-r}.\tag{5}$$

On the other hand, $x_{\omega_i}$ lies in $S^*_\omega$ by assumption. This implies that there exists a unique $\theta_i \in \mathbb{R}^r$ such that

$$x_{\omega_i} = U|_{\omega_i}\theta_i.\tag{6}$$

where $U^*$ is a basis of $S^*$. Substituting (6) in (5) we obtain

$$\{ U|_{\nu_i}^*\theta_i = U|_{\nu_i}U_{\Delta_i}^{-1}U|_{\Delta_i}\theta_i \}_{j=1}^{\ell_i-r}.\tag{7}$$

Recall that $U|_{\Delta_i}^{-1} = U|_{\ell_i}/|U|_{\Delta_i}|$, where $U|_{\ell_i}$ and $|U|_{\Delta_i}$ denote the adjugate and the determinant of $U|_{\ell_i}$. Therefore, we may rewrite (7) as the following set of polynomial equations:

$$\left\{ (|U|_{\Delta_i}|U|_{\nu_i}^* - U|_{\nu_i}U_{\Delta_i}^{-1}U|_{\Delta_i})\theta_i = 0 \right\}_{j=1}^{\ell_i-r}.\tag{8}$$

We conclude that a subspace $S$ with basis $U$ fits $X|_{\Omega}$ if and only if $U$ satisfies (8) for every $i$.

Since every nontrivial subspace has infinitely many bases, even if there is only one $r$-dimensional subspace in $S(X|_{\Omega})$, the variety

$$\{ U \in \mathbb{R}^{d \times x} : \{ (|U|_{\Delta_i}|U|_{\nu_i}^* - U|_{\nu_i}U_{\Delta_i}^{-1}U|_{\Delta_i})\theta_i = 0 \}_{j=1}^{\ell_i-r} \}$$

has infinitely many solutions. Therefore, we will associate a unique $U$ with each subspace as follows. Observe that for every $S \in \text{Gr}_r(r, \mathbb{R}^d)$, we can write $S = \text{span}\{U\}$ for a unique $U$ in the following column echelon form:

$$U = \begin{bmatrix} I \\ V \end{bmatrix}$$

(9)

On the other hand, every $V \in \mathbb{R}^{(d-r) \times r}$ defines a unique $r$-dimensional subspace of $\mathbb{R}^d$, via $\text{span}\{U\}$. Moreover, $\text{span}\{U\}$ will be non-degenerate for almost every $V$, with respect to $\nu_V$: the Lebesgue measure on $\mathbb{R}^{(d-r) \times r}$. Let $\mathbb{R}_{(d-r)\times r}^s \subset \mathbb{R}^{(d-r) \times r}$ denote the set of all $(d-r) \times r$ matrices $V$ whose $\text{span}\{U\}$ is non-degenerate, or equivalently, whose $r \times r$ submatrices of $U$ are full-rank. Then we have a bijection between $\text{Gr}_s(r, \mathbb{R}^d)$ and $\mathbb{R}_{(d-r)\times r}^s$ via $V = \text{span}\{U\}$. It follows that a statement holds for $(\nu_V \times \nu_U)$-almost every pair $(V, \Theta^r)$ if and only if it holds for $(\nu_V \times \nu_U)$-almost every pair $(V^r, \Theta^r)$. We will use these measures interchangeably.

### The Set $\tilde{F}$

Continuing with our analysis, recall that a subspace $S$ with basis $U$ will fit $X_\Omega$ if and only if $U$ satisfies (8) for every $i$. With this in mind, define

$$f_{ij}(V|V^r, \Theta^i) := \left( [U_{\Delta_i} U_{\theta_{ij}}^\nu - U_{\theta_{ij}} U_{\Delta_i}^\nu] \right) \Theta^i,$$

with $U$ and $U^r$ in the column echelon form in (9). We will use $f_{ij}$ as shorthand, with the understanding that $f_{ij}$ is a polynomial in the elements of $V$, and that the elements of $V^r$ and $\Theta^i$ play the role of coefficients.

Furthermore, let

$$\tilde{F}(V|V^r, \Theta^i) := \{f_{ij} \}_{i=1, j=1}^{N, d-r},$$

and use $\tilde{F}(V)$, or simply $\tilde{F}$ as shorthand, with the understanding that $\tilde{F}$ is a set of polynomials in the elements of $V$, and that the elements of $V^r$ and $\Theta^i$ play the role of coefficients. We will also use $\tilde{F} = 0$ as shorthand for $\{f_{ij} = 0\}_{i=1, j=1}^{N, d-r}$.

This way, we may rewrite:

$$S(X_\Omega) = \left\{ \text{span} \begin{bmatrix} I \\ V \end{bmatrix} \in \text{Gr}_s(r, \mathbb{R}^d) : \tilde{F}(V) = 0 \right\}.$$

In general, the affine variety

$$\mathcal{V}(\tilde{F}) := \left\{ V \in \mathbb{R}_{(d-r) \times r}^s : \tilde{F}(V) = 0 \right\}$$

could contain an infinite number of elements. We are interested in conditions that guarantee there is only one or (slightly less demanding) only a finite number. The following lemma states that this will be the case if and only if $r(d-r)$ polynomials in $\tilde{F}$ are algebraically independent.

### Lemma 1

For a.e. $X$, $S(X_\Omega)$ contains at most finitely many subspaces if and only if $r(d-r)$ polynomials in $\tilde{F}$ are algebraically independent.

Proof. By our previous discussion, for a.e. $X$ there are at most finitely many subspaces in $S(X_\Omega)$ if and only if there are at most finitely many points in $\mathcal{V}(\tilde{F})$. We know from algebraic geometry that this will be the case if and only if $\dim \mathcal{V}(\tilde{F}) = 0$ (see, for example, Proposition 6 in Chapter 9, Section 4 of [12]).

Since $\mathcal{V}(\tilde{F}) \subset \mathbb{R}_{(d-r)\times r}^s$, we know that $\dim \mathcal{V}(\tilde{F}) = 0$, then $\tilde{F}$ must contain $r(d-r)$ algebraically independent polynomials (see, for example, Exercise 16 in Chapter 9, Section 6 of [12]).

On the other hand, we know that $\dim \mathcal{V}(\tilde{F}) = 0$ if $r(d-r)$ polynomials in $\tilde{F}$ are a regular sequence (see, for example, Exercise 8 in Chapter 9, Section 4 of [12]).

Finally, since being a regular sequence is an open condition, it follows that for $(\nu_V \times \nu_U)$-almost every pair $(V^r, \Theta^i)$, polynomials in $\tilde{F}$ are algebraically independent if and only if they are a regular sequence (see, for example, Remark 3.4 in [13]).

### Algebraic Independence

By the previous discussion, there are at most finitely many $r$-dimensional subspaces that fit $X_\Omega$ if and only if there is a subset $\tilde{F}$ of $r(d-r)$ polynomials from $\tilde{F}$ that is algebraically independent.

Whether this is the case depends on the supports of the polynomials in $\tilde{F}$, i.e., on $\tilde{\Omega}$: the subset of columns in $\Omega$ corresponding to such polynomials. Lemma 2 shows that the polynomials in $\tilde{F}$ will be algebraically independent if and only if $\tilde{\Omega}$ satisfies the conditions in Theorem 1.

### Lemma 2

For a.e. $X$, the polynomials in $\tilde{F}$ are algebraically independent if and only if $\dim(\Omega^r) = r(m(\Omega^r) - r)$ for some matrix $\Omega^r$ formed with a subset of the columns in $\tilde{\Omega}$.

In order to show this statement, we will require Lemmas 3 and 4 below.

Let $\Omega^r$ be a subset of the columns in $\tilde{\Omega}$, and let $\tilde{\Omega}'$ be the subset of the $n(\Omega^r)$ polynomials in $\tilde{F}$ corresponding to such columns. Notice that $\tilde{\Omega}'$ only involves the variables in $U$ corresponding to the $n(\Omega^r)$ nonzero rows of $\Omega^r$.

Let $n(\Omega^r)$ be the largest number of algebraically independent polynomials in $\tilde{\Omega}'$.

### Lemma 3

For a.e. $X$, $n(\Omega^r) \leq r(m(\Omega^r) - r)$.

Proof. Observe that the column echelon form in (9) was chosen arbitrarily. As a matter of fact, for every permutation of rows $\Pi$ and every $S \in \text{Gr}_s(r, \mathbb{R}^d)$, we may write $S = \text{span}\{U\}$, for a unique $U$ in the following permuted column echelon form:

$$U = \Pi \begin{bmatrix} I \\ V \end{bmatrix}.$$

For example, we could take $\Pi$ to swap the top and bottom blocks in (9), and take $U$ in the following form:

$$U = \begin{bmatrix} I \\ V \end{bmatrix}.$$
Observe that in general, $U$, $V$ and $\tilde{F}$ will be different for each choice of $\Pi$. Nevertheless, the condition $x_{\omega_j} \in S_{\omega_j}$ is invariant to the choice of basis of $S$. This implies that while different choices of $\Pi$ produce different $\tilde{F}$'s, the variety

$$S(X_{\Omega}) = \left\{ \text{span} \left[ \prod \begin{bmatrix} I_V \end{bmatrix} \epsilon \text{Gr}_{s}(r, \mathbb{R}^d) : \tilde{F}(V) = 0 \right] \right\}$$

is the same for every $\Pi$.

This implies that the number of algebraically independent polynomials in $\tilde{F}'$ is invariant to the choice of $\Pi$. Therefore, showing that Lemma 3 holds for one particular $\Pi$ suffices to show that it holds for every $\Pi$.

With this in mind, take $\Pi$ such that $U$ is written with the identity block in the position of $r$ nonzero rows of $\Omega'$.

Since the polynomials in $\tilde{F}'$ only involve the elements of the $m(\Omega')$ rows of $U$ corresponding to the nonzero rows of $\Omega'$, and $U$ has the identity block in the position of $r$ nonzero rows of $\Omega'$, it follows that the polynomials in $\tilde{F}'$ only involve the $r(m(\Omega') - r)$ variables in the $m(\Omega') - r$ corresponding rows of $V$. Furthermore, $\tilde{F}' = 0$ has at least one solution. This implies $\kappa(\Omega') \leq r(m(\Omega') - r)$, as desired. \hfill \Box

We say $\tilde{F}'$ is minimally algebraically dependent if the polynomials in $\tilde{F}'$ are algebraically dependent, but every proper subset of the polynomials in $\tilde{F}'$ is algebraically independent.

**Lemma 4.** For a.e. $X$, if $\tilde{F}'$ is minimally algebraically dependent, then all solutions to $\tilde{F}' = 0$ satisfy $U_{\psi_{ij}} = U_{\psi_{ij}}'$.

In order to prove Lemma 4 we will need the next two lemmas. Let $\omega_{ij}$ index the nonzero entries of the $j^{th}$ column of $\Omega_i$, i.e., $\omega_{ij} := \Delta_i \cup \psi_{ij}$.

**Lemma 5.** Take $\Pi$ such that $U_{\Delta_i} = U_{\Delta_i}' = I$. For a.e. $X$, if $\tilde{F}' = \{\tilde{F}', f_{ij}\}$ is minimally algebraically dependent, then all solutions to $\tilde{F}' = 0$ satisfy $U_{\psi_{ij}} = U_{\psi_{ij}}'$.

The intuition behind this lemma is as follows: suppose for contrapositive that there are infinitely many solutions $x_{\omega_{ij}}$ to $\tilde{F}' = 0$ with $U_{\Delta_i} = I$. Each of these solutions defines a different subspace. Since $\{\tilde{F}', f_{ij}\}$ is minimally algebraically dependent, a.e. solution to $\tilde{F}' = 0$ must fit $x_{\omega_{ij}}$. This will only happen if $x_{\omega_{ij}}$ lies in the intersection of infinitely many $r$-dimensional subspaces, which is at most $(r - 1)$-dimensional. But since $x_{\omega_{ij}}$ is drawn from $S^*$ (an $r$-dimensional subspace), we know that almost surely $x_{\omega_{ij}}$ will not lie in such $(r - 1)$-dimensional subspace.

**Proof.** Suppose that $\tilde{F}' = \{\tilde{F}', f_{ij}\}$ is minimally algebraically dependent, and let $v_j$ denote the $\psi_{ij}$ row of $V$, such that $f_{ij}$ simplifies into

$$f_{ij}(v_j, U_{\Delta_i}, |V^*, \theta_i^*) = \left( U_{\Delta_i} |v_j - v_j U_{\Delta_i} U_{\Delta_i}' \right) \theta_i^* = (v_j - v_j) \theta_i^*.$$

Observe that $f_{ij}$ is the only polynomial in $\tilde{F}_i := \{f_{ij}\}_{i,j \neq 0}$ involving $v_j$. But since $f_{ij}$ involves $v_j$, $\tilde{F}'$ must contain at least one polynomial in $v_j$ (otherwise $\tilde{F}'$ cannot be minimally algebraically dependent). This means that $\tilde{F}''$ contains at least one polynomial $f_{kj}$ involving $v_j$ that is not in $\tilde{F}_i$:

$$f_{kj}(v_j, U_{\Delta_i}, |V^*, \theta_i^*) = \left( U_{\Delta_i} |v_j - v_j U_{\Delta_i} U_{\Delta_i}' \right) \theta_i^*.$$

Since $f_{kj} \notin \tilde{F}_i$, $\theta_i^*$ is independent of $\theta_i^*$, so $(\nu_{V^*} \times \nu_{\theta_i})$-almost surely, $f_{ij} \neq f_{kj}$.

We want to show that if $\tilde{F}'$ is minimally algebraically dependent, then $v_j = v_j^*$ is the only solution to $\tilde{F}' = 0$. So define $v_j = [v_{j1}, v_{j2}]$, and assume for contradiction that there exists a solution to $\tilde{F}'' = 0$ with $v_{j2} = \gamma \neq v_{j2}'$ and $U_{\Delta_i} = I_{\Gamma_k}$, that is also a solution to $\tilde{F}' = 0$.

Next consider the univariate polynomials in $v_{j1}$ evaluated at this solution:

$$g_{ij}(v_{j1}|V^*, \theta_i^*) := f_{ij}(v_{j1}, U_{\Delta_i}, |V^*, \theta_i^*) \bigg|_{v_{j2} = \gamma, U_{\Delta_i} = I_{\Gamma_k}} = f_{ij}(v_{j1}, v_{j2}, U_{\Delta_i}|V^*, \theta_i^*) \bigg|_{v_{j2} = \gamma, U_{\Delta_i} = I_{\Gamma_k}},$$

and observe that since $\{\gamma, I_{\Gamma_k}\}$ are a solution to $\tilde{F}'$, then $g_{ij}$ and $g_{kj}$ must have a common root.

We know from elimination theory that two distinct polynomials $g_{ij}, g_{kj}$ have a common root if and only if their resultant $\text{Res}(g_{ij}, g_{kj})$ is zero (see, for example, Proposition 8 in Chapter 3, Section 5 of [12]).

But $\text{Res}(g_{ij}, g_{kj})$ is a polynomial in the coefficients of $g_{ij}$ and $g_{kj}$. In other words, $\text{Res}(g_{ij}, g_{kj}) = h(V^*, \theta_i^*, \theta_k^*)$ for some nonzero polynomial $h$ in $V^*, \theta_i^*$ and $\theta_k^*$. Therefore, $h \neq 0$ for $(\nu_{V^*} \times \nu_{\theta_i})$-almost every $\{V^*, \theta_i^*\}$ (since the variety defined by $h = 0$ has measure zero). Equivalently, $h = 0$ for a.e. $X$. Since $\text{Res}(g_{ij}, g_{kj}) = 0$, it follows that $g_{ij}$ and $g_{kj}$ do not have a common root $v_{j1}$, which is the desired contradiction.

This will be true for either almost every $\gamma$ in an infinite collection, or for every $\gamma$ in a finite collection. In the first case, we would conclude that $\tilde{F}' = 0$ has infinitely fewer solutions than $\tilde{F}'' = 0$, in contradiction to the minimally algebraically dependent assumption. In the second case, we conclude that $v_{j2}'$ is the only solution to $\tilde{F}' = 0$.

Since $v_{j1}$ was an arbitrary entry of $U_{\omega_{ij}}$, we conclude that for a.e. $X$, if $\tilde{F}'$ is minimally algebraically dependent, then $U_{\psi_{ij}} = U_{\psi_{ij}}'$ is the only solution to $\tilde{F}' = 0$, as desired. \hfill \Box

Define $\{V_t, V_t^\sim\}$ as the partition of the variables involved in the polynomials in $\tilde{F}'$, such that all the variables in $V_t$ are uniquely determined by $\tilde{F}' = 0$.

**Lemma 6.** Suppose $V_t \neq \emptyset$ and that every $f_{ij} \in \tilde{F}'$ is a polynomial in at least one of the variables in $V_t$. Then for a.e. $X$, all the variables involved in $\tilde{F}'$ are uniquely determined by $\tilde{F}' = 0$.

**Proof.** Let $v^c$ be one of the variables in $V_t^c$ and let $f_{ij}$ be a polynomial in $\tilde{F}'$ involving $v^c$. By assumption on $\tilde{F}'$, $f_{ij}$ also involves at least one of the variables in $V_t$, say $v$.

Let $w$ denote the set of all variables involved in $f_{ij}$ except $v$. Observe that $v^c \in w$. This way, $f_{ij}$ is shorthand for $f_{ij}(v, w|V^*, \theta_i^*)$.

We will show that for a.e. $X$, all the variables in $w$ are also uniquely determined by $\tilde{F}' = 0$. 

Suppose there exists a solution to $F' = 0$ with $w = \gamma$, and define the univariate polynomial

$$g(v|V^*, \theta_i') := f_{ij}(v, w|V^*, \theta_i') \big|_{w=\gamma}.$$ 

Now assume for contradiction that there exists another solution to $F' = 0$ with $w \neq \gamma$. Let $w = \gamma'$ be an other solution to $F' = 0$, and define

$$g'(v|V^*, \theta_i') := f_{ij}(v, w|V^*, \theta_i') \big|_{w=\gamma'}.$$ 

We will first show that $g \neq g'$. To see this, recall the definition of $f_{ij}$, and observe that it depends on the choice of $\nu_{ij}$. Nevertheless, it is easy to see that $f_{ij} = 0$ describes the same variety regardless of the choice of $\nu_{ij}$. Intuitively, this means that even though $f_{ij}$ might look different for each choice of $\nu_{ij}$, it really is the same.

Therefore, we may select $\nu_{ij}$ to be the row of $\omega_i$ corresponding to the position of a variable $v$ that takes different values in $\gamma$ and $\gamma'$. This way, a variable with multiple solutions is located in the location of $U_{\nu_{ij}}$. Since $f_{ij}$ is linear in $U_{\nu_{ij}}$, it follows that $g \neq g'$ for $(\nu_v \times \nu_\theta)$-almost every $(V^*, \Theta^*)$.

Now observe that since $v$ is uniquely determined by $F' = 0$, $g$ and $g'$ have a common root, which immediately implies that there are at most finitely many distinct $g'$. Otherwise, $v$ would be a common root to infinitely many distinct polynomials, which $(\nu_v \times \nu_\theta)$-almost surely cannot be the case.

We know from elimination theory that two distinct polynomials $g, g'$ have a common root if and only if their resultant $\text{Res}(g, g')$ is zero (see, for example, Proposition 8 in Chapter 3, Section 5 of [12]).

But $\text{Res}(g, g')$ is a polynomial in the coefficients of $g$ and $g'$. In other words, $\text{Res}(g, g') = h(V^*, \Theta_i')$ for some nonzero polynomial $h$ in $V^*$ and $\Theta_i'$. Therefore, $h \neq 0$ for $(\nu_v \times \nu_\theta)$-almost every $(V^*, \Theta^*)$ (since the variety defined by $h = 0$ has measure zero). Equivalently, $h \neq 0$ for a.e. $X$.

Since $\text{Res}(g, g') \neq 0$, it follows that $g$ and $g'$ do not have a common root $v$, which is the desired contradiction. This shows that for a.e. $X$, all the variables in $w$ (including $v^c$) are uniquely determined by $F' = 0$.

Since $v^c$ was an arbitrary element in $V^c_i$, we conclude that all the variables in $V^c_i$ are also uniquely determined by $F' = 0$.

With this, we are now ready to present the proofs of Lemma 4, Lemma 2 and Theorem 1.

**Proof.** (Lemma 4) By the same arguments as in Lemma 3, whether $F'$ is minimally algebraically dependent is invariant to any permutation $\Pi$ of the rows of the column echelon form in (9). Therefore, showing that Lemma 4 holds for one particular choice of $\Pi$ suffices to show it holds for every $\Pi$.

With this in mind, suppose $F' = \{F''', f_{ij}\}$ is minimally algebraically dependent. Take $\Pi$ such that $U$ and $U^*$ are written in the column echelon form in (9) with the identity block in the rows indexed by $\Delta_i$, and let $v_{ij}$ denote the row of $V$ corresponding to $U_{v_{ij}}$, such that

$$U_{v_{ij}} = \begin{bmatrix} I \\ v_{ij} \end{bmatrix}.$$ 

We know by Lemma 5 that $v_{ij}$ is uniquely determined by $F' = 0$. We will now iteratively use Lemma 6 to show that all the variables in $F''$ (which are the same as the variables in $F'''$) are also uniquely determined by $F' = 0$. This will imply that all the variables in $F'''$ are finitely determined by $F'' = 0$, and that $F'''$ contains the same number of polynomials, $n(\Omega'')$, as variables, $r(m(\Omega') - r)$, which is the desired conclusion.

First observe that since $v_{ij}$ is finitely determined by $F'' = 0$, $F'''$ must contain at least $r$ polynomials in $v_{ij}$. Denote these polynomials by $U_1 \subset F'''$.

We will proceed inductively, indexed by $t \geq 1$. First, set $t = 1$ and define $V_1 = \{v_{ij}\}$. We showed above that the variables in $V_1$ are uniquely determined by $F'_1 = 0$. Suppose that $F'_1$ involves some variables other than those in $V_1$. Note that every polynomial in $F'_1$ involves at least one of the variables in $V_1$. Define $V_2$ to be the set of all variables involved in $F'_1$. By Lemma 6, all the variables in $V_2$ are uniquely determined by $F'_1 = 0$.

We will now proceed inductively. For any $t \geq 2$, let $V_t$ be a set of $n_t$ variables in $V$. Assume that all the variables in $V_t$ are uniquely determined by $F'_t = 0$. Since $\text{dim} V(\Omega'') = \text{dim} V(\Omega')$, it follows that all the variables in $V_t$ are finitely determined by $F'' = 0$. It follows that $F''$ must contain at least $n_t$ algebraically independent polynomials, each involving at least one of the variables in $V_t$. Let $F'_t$ be this set of polynomials. Suppose $F'_t$ involves some variables other than $V_t$. Define $V_{t+1}$ to be the set of all variables involved in $F'_t$. By Lemma 6, all the variables in $V_{t+1}$ are uniquely determined by $F_t' = 0$.

Since this is true for every $t$, and there are finitely many variables, this process must terminate at some finite step $T$, at which point $F'_T$ is a set of $n_T$ algebraically independent polynomials in $n_T$ variables.

This means that all the variables in $F'_T$ are finitely determined by $F'_T = 0$, and since $f_{ij}$ only involves a subset of the variables in $F'_T$, it follows that the polynomials in $\{f_{ij}\} \subset F'$ are algebraically dependent. Furthermore, since $F'$ is minimally algebraically dependent by assumption, we have that $F'_T = F''$.

Finally, observe that $F''$ contains $n(\Omega'')$ polynomials in $r(m(\Omega') - r)$ variables. Since $F'' = F'_T$, and $F'_T$ has $n_T$ polynomials in $n_T$ variables, it follows that $n(\Omega'') = r(m(\Omega') - r)$, as desired.

**Proof.** (Lemma 2)

$(\Rightarrow)$ Suppose $F'$ is minimally algebraically dependent. By Lemma 4, $n(\Omega') = r(m(\Omega') - r) + 1 > r(m(\Omega') - r)$, and we have the first implication.
Suppose there exists an $\Omega'$ with $n(\Omega') > r(m(\Omega') - r)$. By Lemma 3, $n(\Omega') > n(\Omega')$, which implies the polynomials in $\mathcal{F}'$, and hence $\mathcal{F}$, are algebraically dependent.

Proof. (Theorem 1)

$\Rightarrow$ Suppose for contrapositive that for every $\tilde{\Omega}$ there exists an $\Omega'$ formed with a subset of its columns such that $m(\Omega') < n(\Omega')/r + r$. Lemma 2 implies that the polynomials in $\mathcal{F}'$, and hence $\mathcal{F}$, are algebraically dependent. It follows by Lemma 1 that there are infinitely many subspaces in $\Sigma(X_{\Omega})$.

$\Leftarrow$ Suppose every $\Omega'$ formed with a subset of the columns in $\tilde{\Omega}$ satisfies $m(\Omega') \geq n(\Omega')/r + r$, including $\tilde{\Omega}$. By Lemma 2, the $r(d-r)$ polynomials in $\mathcal{F}$ are algebraically independent. It follows by Lemma 1 that there are at most finitely many subspaces in $\Sigma(X_{\Omega})$, hence at most finitely many rank-$r$ completions of $X_{\Omega}$.

IV. UNIQUE COMPLETABILITY

In this section we give the proof of Theorem 2. Similar to $\Omega_i$ and $\tilde{\Omega}$, define

$$X_{\Omega_i} := \begin{bmatrix} \mathbf{x}_{\omega_i} & \cdots & \mathbf{x}_{\omega_i} \end{bmatrix}_{r}$$

where empty spaces represent missing values. Then concatenate these matrices to obtain $X_{\tilde{\Omega}} := [X_{\Omega_1} \cdots X_{\Omega_n}]$.

We will use $\mathbf{x}_{\tilde{\Omega}_i}$ and $\mathbf{x}_{\tilde{\Omega}}$ to denote the $d \times r(d-r)$ and $d \times (d-r)$ submatrices of $X_{\tilde{\Omega}}$ corresponding to $\tilde{\Omega}$ and $\Omega$. In addition, let $\tilde{x}_{\omega_i}$ and $\mathbf{x}_{\tilde{\omega}_i}$ denote the $d \times r(d-r)$ submatrices of $\tilde{\Omega}$ and $\Omega$.

In order to prove Theorem 2, we will require Theorem 1 in [14], which we state here as the following lemma, with some minor adaptations to our context.

Lemma 7. Suppose $\tilde{\Omega}$ is a $d \times (d-r)$ matrix with binary entries for which (ii) holds and let $S$ in $\text{Gr}(r, \mathbb{R}^d)$. Then for $\nu_{\mathcal{G}}$-almost every $S^*$, \{ $S_{\omega_i} = S_{\omega_i}^*$ \} if and only if $S = S^*$.

With this, we are ready to prove the proof of Theorem 2.

Proof. (Theorem 2) Suppose $\tilde{\Omega}$ contains two disjoint matrices $\tilde{\Omega}$ and $\Omega$ satisfying the conditions of Theorem 2.

Since $\tilde{\Omega}$ satisfies (ii), by Theorem 1 there are at most finitely many $r$-dimensional subspaces that fit $X_{\tilde{\Omega}}$. Equivalently, the set $\mathcal{F}$, containing the $r(d-r)$ polynomials defined by the columns in $X_{\tilde{\Omega}}$, is algebraically independent. Let $f_i$ be the polynomial defined by $\mathbf{x}_{\omega_i}$. It follows that the set \{ $\mathcal{F}, f_i$ \} is algebraically dependent. Let $\mathcal{F}''$ be a subset of the polynomials in $\mathcal{F}$, such that $\mathcal{F}' = \{ \mathcal{F}, f_i \}$ is minimally algebraically dependent. Then any subspace $S$ with basis $U$ that fits $\mathbf{x}_{\omega_i}$ must satisfy $\mathcal{F}' = 0$, implying by Lemma 5 that $U_{\omega_i} = U_{\omega_i}'$.

Therefore, every $S$ that fits both $X_{\Omega_i}$ and $X_{\tilde{\Omega}}$ must satisfy $\{ S_{\omega_i} = S_{\omega_i}^* \}$. Since $\Omega_\omega$ satisfies (ii), it follows by Lemma 7 that $S = S^*$.

In Section II we mentioned that there are cases where $\tilde{N} = r(d-r)$ is sufficient for unique completability. The next result states that this is indeed the case if $r = 1$.

Proposition 1. If $r = 1$, finite completability is equivalent to unique completability.

Proof. Assume $r = 1$. Then $U_{\omega_i}$ and $U_{\omega_{ij}}$ are scalars, so $f_{ij}$ simplifies into:

$$f_{ij} = \left( U_{\omega_i} U_{\omega_{ij}} - U_{\omega_{ij}} U_{\omega_j}^\prime \right) \theta_i^\prime.$$

This implies that $\mathcal{F} = 0$ is a system of linear equations, hence if it has finitely many solutions, it has only one.

In Section II we also mentioned that in general, $\tilde{N} > r(d-r)$ is necessary for unique completability. We would like to close this section with an example where this is the case.

Example 3. Consider $d = 4$ and $r = 2$, such that $\tilde{N} = r(d-r) = 4$. Let

$$\Omega = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

It is easy to see that $\tilde{\Omega} = \Omega = \Omega$ satisfies the conditions of Theorem 1. One may also verify (for example, solving explicitly $\mathcal{F}(V) = 0$) that for a.e. $X$ there exist two subspaces that fit $X_{\Omega}$.

As a matter of fact, this will also be the case for any permutation of the rows and columns of this matrix. One may construct similar sampleings with the same property for larger $d$ and $r$. All this to say that this is not a singular pathological example; there are many sampleings that cannot be uniquely recovered with only $\tilde{N} = r(d-r)$.

V. RANDOM SAMPLING PATTERNS

In this section we present the proof of Theorem 3. To do so, we will use the following lemma, which is an additional sufficient condition for finite completability. This useful result also shows the tight relation between the conditions for finite completability and the condition in (ii).

Lemma 8. $\Omega$ satisfies (i) if it contains disjoint matrices $\{ \Omega_{\omega_i} \}_{i=1}^m$, each of size $d \times (d-r)$, such that (ii) holds for every $\Omega_{\omega_i}$.

Proof. Suppose $\tilde{\Omega}$ contains disjoint matrices $\{ \Omega_{\omega_i} \}_{i=1}^m$ satisfying the conditions of Lemma 8. Let $\mathbf{x}'$ be a matrix formed with a subset of the columns in $\tilde{\Omega}$. Then $\mathcal{F}' = \{ \Omega_{\omega_i}^\prime \}$ for some matrices $\{ \Omega_{\omega_i}^\prime \}_{i=1}^m$ formed with subsets of the columns in $\{ \Omega_{\omega_i} \}_{i=1}^m$. 


It follows that
\[ n(\Omega') = \sum_{\tau=1}^{r} n(\Omega'_\tau) \leq \sum_{\tau=1}^{r} \max_{\tau} n(\Omega'_\tau). \]
Assume without loss of generality that this maximum is achieved when \( \tau = 1 \). Then
\[ n(\Omega') \leq r n(\Omega'_1) \leq r (m(\Omega'_1) - r) \leq r (m(\Omega') - r), \]
where the last two inequalities follow because (2) holds for every \( \Omega'_\tau \) by assumption, and because \( m(\Omega') \geq m(\Omega'_1) \) for every \( \tau \).
Since \( \Omega' \) was arbitrary, we conclude that (1) holds for every matrix \( \Omega' \) formed with a subset of the columns in \( \bar{\Omega} \).
\( \square \)

**Example 4.** The partition \( \Omega = [\bar{\Omega}_1 \cdots \bar{\Omega}_r] \) in Example 1 satisfies the conditions in Lemma 8.

The following lemma shows that (ii) is satisfied with high probability under uniform random sampling schemes with only \( O(\max(r, \log d)) \) samples per column. The proof of Theorem 3 follows directly by applying this result and a union bound.

**Lemma 9.** Let the assumptions of Theorem 3 hold, and let \( \bar{\Omega} \) be a matrix formed with \( d - r \) columns of \( \Omega \). With probability at least \( 1 - \frac{\ell}{d} \), \( \bar{\Omega} \) will satisfy (ii).

**Proof.** Let \( \mathcal{E} \) be the event that \( m(\Omega') < n(\Omega') + r \) for some matrix \( \Omega' \) formed with a subset of the columns in \( \bar{\Omega} \). It is easy to see that this will only occur if there is a matrix \( \Omega' \) formed with \( n \) columns of \( \bar{\Omega} \) that has all its nonzero entries in the same \( n + r - 1 \) rows. Let \( \mathcal{E}_n \) denote the event that the matrix formed with the first \( n \) columns from \( \bar{\Omega} \) has all its nonzero entries in the first \( n + r - 1 \) rows. Then
\[
P(\mathcal{E}) \leq \sum_{n=1}^{d-r} \binom{d-r}{n} \binom{d}{n} P(\mathcal{E}_n) \tag{10}
\]
If each column of \( \bar{\Omega} \) contains at least \( \ell \) nonzero entries, distributed uniformly and independently at random with \( \ell \) as in \((3)\), it is easy to see that \( P(\mathcal{E}_n) = 0 \) for \( n \leq \ell - r \), and for \( \ell - r < n \leq d - r \),
\[
P(\mathcal{E}_n) \leq \left( \frac{\binom{n+r-1}{\ell}}{\binom{d}{\ell}} \right)^n < \left( \frac{n+r-1}{d} \right)^n .
\]
Since \( \binom{d-r}{n} < \binom{d}{n-r+1} \), continuing with \((10)\) we obtain:
\[
P(\mathcal{E}) < \sum_{n=\ell-r+1}^{d} \binom{d}{n} \binom{d}{n+r-1} (n+r-1)^n 
< \sum_{n=\ell-r+1}^{d} \binom{d}{n} \frac{(n+r-1)^n}{(n+r-1)^n} 
< \frac{d}{\ell} \left( \frac{n}{d} \right)^{\ell(n-r+1)} \tag{11}
\]
\[
+ \frac{d}{\ell} \left( \frac{d-n}{d} \right)^{\ell(d-r-1)} \tag{12}
\]
For the terms in \((11)\), write
\[
\left( \frac{d}{n} \right)^{\ell(n-r+1)} < \left( \frac{d}{n} \right)^{2n} \left( \frac{n}{d} \right)^{\ell(n-r+1)} . \tag{13}
\]
Since \( n \geq \ell \geq 2r \),
\[
(13) < \left( \frac{d}{n} \right)^{2n} \left( \frac{n}{d} \right)^{\ell(n-r+1)} = e^{2n} \left( \frac{n}{d} \right)^{\ell(n-r+1)} , \tag{14}
\]
and since \( n \leq \frac{d}{\ell} \),
\[
(14) \leq e^{2n} \left( \frac{1}{2} \right)^{\ell(n-r+1)} = \left( e^{2n} \cdot 2^{-\ell(n-r+1)} \right) < \frac{\epsilon}{d^2} , \tag{15}
\]
where the last step follows because \( \ell > 2 \log_2 \left( \frac{d^2}{\ell} \right) + 4 \).
For the terms in \((12)\), write
\[
\left( \frac{d}{n} \right)^{\ell(n-r+1)} < \left( \frac{d}{n} \right)^{2n} \left( \frac{n}{d} \right)^{\ell(n-r+1)} . \tag{16}
\]
In this case, since \( 1 \leq n \leq \frac{d}{\ell} \) and \( r \leq \frac{d}{\ell} \), we have
\[
(16) < \left( \frac{d}{n} \right)^{2n} \left( \frac{d-n}{d} \right)^{\ell(d-r+1)} = \left( \frac{d}{n} \right)^{2n} \left( 1 - \frac{n}{d} \right)^{\ell} \leq \left( \frac{d}{n} \right)^{2n} \left( e^{-n} \right)^{\ell} , \tag{17}
\]
which we may rewrite as
\[
\left( e^{2\log d} \right)^n \left( e^2 \right)^n \left( e^{-n} \right)^{\ell} = \left( e^{2\log d + 2 - \ell} \right)^n < \frac{\epsilon}{d^2} , \tag{17}
\]
where the last step follows because \( \ell > 3 \log (\frac{d^2}{\ell}) + 6 \log d + 6 \).

Substituting \((15)\) and \((17)\) in \((11)\) and \((12)\), we have that \( P(\mathcal{E}) < \frac{\epsilon}{d^2} \). \( \square \)

**References**


