Ballooning filament growth in the intermediate nonlinear regime

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Abstract

A theory is developed for the description of ballooning instability in the intermediate nonlinear regime for general magnetic configurations including toroidal systems such as tokamaks. The evolution equations for the plasma filament growth induced by the ballooning instability are derived accounting for the dominant nonlinear effects in an ideal MHD description. The intermediate nonlinear regime of ballooning modes is defined by the ordering that the plasma filament displacement across the magnetic surface is comparable to the linear mode width in the same direction. In the tokamak case, this regime could become particularly relevant for a transport barrier as the width of that barrier (or pedestal) region approaches the mode width of the dominant ballooning mode. A remarkable feature of the nonlinear ballooning equations is that solutions of the associated local linear ballooning mode equations continue to be valid solutions into the intermediate nonlinear regime. The filament growth equations for the intermediate nonlinear ballooning regime may be applicable to the precursor and pre-collapse phase of edge localized modes (ELMs) observed in both simulations and experiments.
I. INTRODUCTION

Filamentary structures and their localization in the unfavorable curvature region of the tokamak edge have been routinely observed during periods of edge localized modes (ELMs) in recent MAST experiments [1, 2] and extended MHD simulations [3]. This indicates that the ballooning instability properties of the pedestal region continues to play a dominant role in determining the nonlinear temporal and spatial structures of ELMs. Thus it may be possible to understand the dynamics of the ELM filaments in terms of the nonlinear properties of the ballooning instability. In this work, we develop a theory for the nonlinear evolution of ballooning instabilities using an ideal MHD model for general toroidal magnetic configurations.

A typical ELM event in a tokamak would evolve through a sequence of phases with different time scales [2]. The precursor phase has a time scale about $100\mu s \sim 100 - 1000\tau_A$, where the Alfvén time scale $\tau_A$ is defined by $Rq/u_A \sim \sqrt{RL_p/c_s}$, with $R$ being the major radius, $q$ the safety factor, $L_p$ the pedestal width (or pressure gradient scale length), $u_A$ the local Alfvén speed, and $c_s$ the sound speed. This phase of ELM is associated with the linear to early nonlinear MHD activity. The precursor phase terminates at the onset of the collapse phase, during which the edge diagnostic signal levels rapidly increase and drop in a time scale of about $10\mu s \sim 10 - 100\tau_A$. The recovery phase links the end of the collapse phase to the precursor phase of the next ELM cycle. This phase takes place on a transport time scale $\tau_E \sim 10\text{ms}$ which allows the edge pedestal to recover its height in H-mode.

Different phases of ELMs may relate to different linear and nonlinear regimes of ballooning instability. To describe the different nonlinear phases, we introduce two small parameters given by

\[ n^{-1} = \frac{k_\parallel}{k_\perp} \ll 1, \quad \varepsilon_\xi = \frac{|\xi|}{L_{eq}} \ll 1. \]

Here, $k_\parallel$ and $k_\perp$ are the dominant wavevectors of the perturbation parallel to and perpendicular to the equilibrium magnetic field lines, respectively; $\xi$ is the ideal MHD plasma displacement produced by instability, and $L_{eq}$ is the equilibrium scale length (which is used as the normalization length later so that $L_{eq} = 1$). The first small parameter $n^{-1}$ can be used to measure the three intrinsic spatial scales that distinguish the structure of a linear ballooning mode. They are the mode width in the direction parallel to equilibrium magnetic field, $\lambda_\parallel \sim k_\parallel^{-1}$; the mode width in the direction across magnetic flux surface, $\lambda_\perp$; and the
mode width in the most rapidly varying direction perpendicular to the equilibrium magnetic field, \( \lambda_\perp \sim k_\perp^{-1} \). If \( L_{eq} \) is chosen to be the same order of \( \lambda_\parallel \), as in \( \lambda_\parallel \sim L_{eq} \sim 1 \), then we have \( \lambda_\perp \sim n^{-1/2} \), and \( \lambda_\perp \sim n^{-1} \).

The linear structure and growth rate of ballooning instability can be determined using an asymptotic expansion of the linearized ideal MHD equation in terms of \( n^{-1} \) [4–6]. The mode structure in the fastest varying direction perpendicular to the field line is given by \( n^{-1} \), which is the scale of the dominant wavelength. At lowest order in \( n^{-1} \), the ballooning mode is described by two coupled one-dimensional ordinary differential equations along each field line, which together with proper boundary conditions, determines the local eigenfrequency or growth rate as well as the local mode structure along the equilibrium magnetic field as a function of magnetic flux surface, field line, and radial wavenumber. At higher order in \( n^{-1} \), a global eigenmode equation, the envelope equation, which uses information from the local mode calculations, governs the global growth rate and mode structure across the magnetic surface. In axisymmetric equilibria, the global growth rate is given by the most unstable value of the local growth rate with stabilizing corrections of order \( n^{-1} \). As shown earlier [7–11] and in this work, the properties of linear ballooning instability are crucial to the construction and understanding of the theory of nonlinear ballooning instability.

The perturbation amplitude of the nonlinear ballooning mode, measured by \( \varepsilon_\xi \) or \( |\xi| \), can be compared to those characteristic spatial scales of its linear mode structure. In the early nonlinear regime, which is defined by the ordering \( |\xi| \sim \lambda_\perp \sim n^{-1} \), the filament scale across the magnetic flux surface is comparable to the mode width in the most rapidly oscillating direction [7–9]. In this regime, \( |\xi| \ll \lambda_\perp \), nonlinear convection across the flux surface is small relative to the mode width in that direction, and nonlinearities modify only the radial envelope equation describing mode evolution across the magnetic surface. As the mode continues to grow, it enters the intermediate nonlinear regime, in which \( |\xi| \sim \lambda_\perp \sim n^{-1/2} \); the plasma displacement across magnetic flux surface becomes of the same order as the mode width in that direction [10, 11]. In this regime, effects due to convection and compression are no longer small. Nonlinearities due to convection and compression, together with nonlinear line-bending forces, directly modify the “local” mode evolution along the magnetic field line. In the late nonlinear regime, the ballooning filament growth may exceed the scale of the pedestal width and result in the collapse of the pedestal. Eventually, these ballooning filaments could detach from edge plasma and propagate into the scrape-off-layer.
region, as indicated from recent experiment [2]. In this work, we emphasize the physics of the intermediate nonlinear phase and delay discussions of the late nonlinear regime for subsequent work.

It is conceivable that the linear to early nonlinear regime of the ballooning instability of the pedestal may mostly correspond to the precursor phase of ELMs since the onset of the ELMs have been consistently correlated to the breaching of linear stability boundary of the peeling-ballooning mode [12, 13]. Earlier theory attempted to explain the collapse onset phase of ELMs by invoking a finite time-like singularity associated with the early nonlinear ballooning instability of a marginally unstable configuration (“Cowley-Artun regime”) [7–9]. Such a scenario, however, has yet to be confirmed by direct MHD simulations, probably due to the rather limited range of validity for that regime. In contrast, there is a good agreement between the prediction of intermediate nonlinear ballooning mode equation and results from direct MHD simulation for the case of a line-tied \( g \)-mode [11]. It is likely that the intermediate nonlinear regime may better characterize the transition from the precursor phase to the collapse onset of an ELM. This regime could become particularly relevant for a transport barrier as the width of that barrier (or pedestal) region approaches the mode width of the dominant ballooning mode.

In this work, the theory for the intermediate nonlinear ballooning instability in general and toroidal configurations is developed. We first lay out the general formulation in Clebsch coordinate in Sec. II, and in a tokamak flux coordinate in Sec. III. To facilitate analytical and numerical analysis, we further consider the circular shaped tokamak in large aspect ratio limit, and formulate our theory in Shafranov coordinates in Sec. IV. The relevant equations for the intermediate nonlinear regime of ballooning instability in an \( s - \alpha \) model equilibrium are given. Finally we conclude with a summary and discussion in Sec. V.

II. BALLOONING FILAMENT EQUATIONS IN CLEBSCH COORDINATES

The nonlinear theory of ballooning mode can be conveniently developed in the Lagrangian formulation of the ideal MHD model [14]

\[
\frac{\rho_0}{J} \nabla_0 \mathbf{r} \cdot \frac{\partial^2 \xi}{\partial t^2} = -\nabla_0 \left[ \frac{\rho_0}{J^2} \right] + \nabla_0 \mathbf{r} \cdot \left[ \frac{\mathbf{B}_0}{J} \nabla_0 \left( \frac{\mathbf{B}_0}{J} \nabla_0 \mathbf{r} \right) \right] + \frac{\rho_0}{J} \nabla_0 \mathbf{r} \cdot \mathbf{g} \tag{2}
\]
where
\[ r(r_0, t) = r_0 + \xi(r_0, t), \quad \nabla_0 = \frac{\partial}{\partial r_0}, \quad J(r_0, t) = |\nabla_0 r|. \quad (3) \]

Here, \( r_0 \) denotes the initial location of each plasma element in the equilibrium, \( \xi \) is the plasma displacement from the initial location, and \( J(r_0, t) \) is the Jacobian for the Lagrangian transformation from \( r_0 \) to \( r(r_0, t) \); \( \rho_0 \), \( p_0 \), and \( B_0 \) are the equilibrium mass density, pressure, and magnetic field, respectively. For the sake of completeness, we include the effects of gravity \( g \) to facilitate connection to our previous work on line-tied \( g \) mode [10, 11]. For applications to toroidal confinement systems, these terms can be dropped. We first consider a general magnetic configuration
\[ B_0 = \nabla_0 \Psi_0 \times \nabla_0 \alpha_0 \quad (4) \]

in a nonorthogonal Clebsch coordinate system \((\Psi_0, \alpha_0, l_0)\), where \( \Psi_0 \) is the magnetic flux label, \( \alpha_0 \) the field line label, and \( l_0 \) the measure of field line length. The corresponding coordinate Jacobian is given by \( (\nabla_0 \Psi_0 \times \nabla_0 \alpha_0 \cdot \nabla_0 l_0)^{-1} = |B_0|^{-1} \). The plasma filament size due to ballooning instability can be quantified by the plasma displacement \( \xi \), which is expanded in terms of \( n^{-1} \) and \( \varepsilon \) as follows
\[ \xi(\sqrt{n} \Psi_0, n \alpha_0, l_0, t) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \varepsilon_i \varepsilon_j^{n^{-1/2}} \left( e_{\Psi} \xi_{i,j}^{\Psi} + \frac{e_{\alpha}}{\sqrt{n}} \xi_{i,j}^{\alpha} + e_{l} \xi_{i,j}^{l} \right) \quad (5) \]
where \( e_{\Psi} = B_0^{-1} \nabla_0 \alpha_0 \times \nabla_0 l_0 \), \( e_{\alpha} = B_0^{-1} \nabla_0 l_0 \times \nabla_0 \Psi_0 \), \( e_{l} = B_0^{-1} B_0 = b \), and \( B_0 = |B_0| \). Here and subsequently we drop the subscript “0” in the equilibrium MHD fields \( \rho_0 \), \( p_0 \), and \( B_0 \) for convenience. The plasma displacement \( \xi \) and the Lagrangian Jacobian \( J \) are functions of the normalized coordinates \((\Psi, \alpha, l)\), where \( \Psi = \sqrt{n} \Psi_0 \), \( \alpha = n \alpha_0 \), \( l = l_0 \).

The intermediate nonlinear regime is defined by the ordering \( \varepsilon \sim O(n^{-1/2}) \) [10, 11]. In this regime the plasma displacement \( \xi \) and the Lagrangian Jacobian \( J \) are expanded as a single series of \( n^{-1} \)
\[ \xi(\sqrt{n} \Psi_0, n \alpha_0, l_0, t) = \sum_{j=1}^{\infty} n^{-j/2} \left( e_{\Psi} \xi_{j}^{\Psi} + \frac{e_{\alpha}}{\sqrt{n}} \xi_{j+1}^{\alpha} + e_{l} \xi_{j}^{l} \right), \quad (6) \]
\[ J(\sqrt{n} \Psi_0, n \alpha_0, l_0, t) = 1 + J_0 + \sum_{j=1}^{\infty} n^{-j/2} J_j. \quad (7) \]
Setting \( \xi = 0 \) in Eq.(2) gives the equilibrium relation
\[ \nabla_0 (p + \frac{B^2}{2}) = B \cdot \nabla_0 B + \rho g. \quad (8) \]
At third order, nonlinearities enter the equation for $\xi$ where $\kappa$
the lowest order equation for $\xi$
The lowest order expansions of the full MHD equation (2) yield the equations

$$O(\sqrt{n}) : \left[ \frac{B^2}{(1 + J_0)^3} + \frac{\gamma \rho}{(1 + J_0)^{\gamma + 1}} \right] \partial_\psi J_0 = 0,$$

$$O(n) : \left[ \frac{B^2}{(1 + J_0)^3} + \frac{\gamma \rho}{(1 + J_0)^{\gamma + 1}} \right] \partial_\alpha J_0 = 0,$$

$$O(1) : -\rho \partial_t \left[ \frac{1}{1 + J_0} \right] + \left[ \frac{1}{1 + J_0} - \frac{1}{(1 + J_0)^\gamma} \right] \partial_\psi \rho = 0.$$

This indicates, to leading order, the incompressible condition, $J_0 = \text{const} = 0$.

For convenience, a new set of basis vectors are introduced [8]

$$e_\perp = e_\psi \cdot (I - \mathbf{b} \mathbf{b}) = \frac{\nabla_0 \alpha_0 \times \mathbf{B}}{B^2}, \quad e_\alpha = e_\alpha \cdot (I - \mathbf{b} \mathbf{b}) = \frac{\mathbf{B} \times \nabla_0 \psi_0}{B^2}$$

so that

$$\xi = e_\perp \xi^\psi + e_\alpha \xi^\alpha + \mathbf{B}\xi^\parallel.$$  

At next order, perpendicular and parallel force balance yield, respectively

$$(\gamma \rho + B^2)J_2 - \mathbf{B} \cdot \nabla_0 (\mathbf{B} \cdot \xi_2) + 2\mathbf{B} \cdot \nabla_0 \mathbf{B} \cdot \xi_2 + \rho \mathbf{g} \cdot \xi_2 = F_2^\perp (l, t),$$

$$\rho B \partial^2_\parallel \xi_2 = \partial_\psi (\gamma \rho J_2) - \rho \mathbf{g} \cdot \mathbf{b} J_2 + \rho \mathbf{g} \cdot \partial_\psi \xi_2.$$  

With the generalized natural boundary condition $F_2^\perp (l, t) = 0$, we have

$$J_2 = \frac{B_{\perp}^2}{1 + \gamma \beta} \left[ B^2 \partial_\psi \xi_2^\parallel - \rho \mathbf{g} \cdot \mathbf{B} \xi_2^\parallel - e_\perp \cdot (2B^2 \kappa + \rho \mathbf{g}) \xi_2^\psi \right].$$  

where $\kappa = \mathbf{b} \cdot \nabla_0 \mathbf{b}$ is the magnetic curvature. From the parallel force balance (15) we obtain the lowest order equation for $\xi^\parallel$ which remains linear formally [8]

$$\rho B^2 \partial^2_\parallel \xi_2^\parallel = \mathcal{L}_\parallel (\xi_2^\psi, \xi_2^\parallel),$$

$$\mathcal{L}_\parallel (\xi_2^\psi, \xi_2^\parallel) = B \partial_\psi \left[ \frac{\gamma \rho}{1 + \gamma \beta} \left( \partial_\psi \xi_2^\parallel - 2e_\perp \cdot \kappa \xi_2^\psi + \frac{\rho \mathbf{g} \cdot e_\perp}{\gamma \rho} \xi_2^\psi \right) \right]$$

$$+ \left[ \frac{(\rho \mathbf{g} \cdot \mathbf{b})^2}{1 + \gamma \beta} + \rho B \partial_\psi (\mathbf{g} \cdot \mathbf{B}) - B \partial_\psi \left( \frac{\gamma \beta}{1 + \gamma \beta} \rho \mathbf{g} \cdot \mathbf{B} \right) \right] \xi_2^\parallel$$

$$+ \left[ \frac{\rho \mathbf{g} \cdot \mathbf{b}}{\gamma \rho + B^2} e_\perp \cdot (2B^2 \kappa + \rho \mathbf{g}) - e_\perp \cdot \mathbf{g} B \partial_\psi \rho \right] \xi_2^\psi.$$  

At third order, nonlinearities enter the equation for $\xi^\psi$:

$$\rho \left\{ |e_\perp|^2 \partial_\alpha \partial^2_\parallel \xi_2^\psi + |e_\perp|^2 [\xi_2^\psi, \partial^2_\parallel \xi_2^\psi] + B^2 [\xi_2^\parallel, \partial^2_\parallel \xi_2^\parallel] \right\}$$

$$= \partial_\alpha \mathcal{L}_\perp (\xi_2^\psi, \xi_2^\parallel) + [\xi_2^\psi, B \partial_\psi (B \partial_\psi \xi_2^\parallel) - B B \partial_\psi J_2 - (\rho \mathbf{g} + 2B \partial_\psi \mathbf{B}) J_2^\parallel].$$  

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where \([A, B] \equiv \partial_\psi A \partial_\psi B - \partial_\alpha A \partial_\psi B, \) \([A; B] \equiv \partial_\psi A \cdot \partial_\psi B - \partial_\alpha A \cdot \partial_\psi B,\) and

\[
\mathcal{L}_\perp(\xi^\psi_2, \xi^\parallel_2) \equiv B \partial_l(|\mathbf{e}_\perp|^2B \partial_\xi^\psi_2) + [2\mathbf{e}_\perp \cdot \kappa \mathbf{e}_\perp \cdot (\nabla_0 \rho - \rho g) - \mathbf{e}_\perp \cdot \mathbf{g} \mathbf{e}_\perp \cdot \nabla_0 \rho] \xi^\psi_2
\]

\[+ \frac{2 \gamma p e_\perp \cdot \kappa}{1 + \gamma \beta} \left( B \partial_l \xi^\parallel_2 - 2 \mathbf{e}_\perp \cdot \kappa \xi^\psi_2 + \frac{\rho g \cdot \mathbf{e}_\perp}{\gamma p} \xi^\psi_2 \right)\]

\[- \frac{\rho g \cdot \mathbf{e}_\perp}{1 + \gamma \beta} \left( B \partial_l \xi^\parallel_2 - 2 \mathbf{e}_\perp \cdot \kappa \xi^\psi_2 - \frac{\rho g \cdot \mathbf{e}_\perp}{B^2} \xi^\psi_2 \right)\]

\[+ \left[ \frac{\rho g \cdot \mathbf{B}}{B^2(1 + \gamma \beta)} \mathbf{e}_\perp \cdot (2B^2 \mathbf{\kappa} + \rho \mathbf{g}) - \mathbf{e}_\perp \cdot \mathbf{g} \mathbf{B} \cdot \nabla_0 \rho \right] \xi^\parallel_2. \quad (20)\]

The linear operators \(\mathcal{L}_\perp\) and \(\mathcal{L}_\parallel\) were obtained earlier in [8], which in absence of gravity reduce to those in conventional linear ballooning theory [4–6, 15]. Equation (19) is further simplified as

\[
\rho |\mathbf{e}_\perp|^2(\partial_\alpha \partial^2 \xi^\psi_2 + [\xi^\psi_2, \partial^2 \xi^\parallel_2]) = \partial_\alpha \mathcal{L}_\perp(\xi^\psi_2, \xi^\parallel_2) + |\mathbf{e}_\perp|^2[B \partial_l(B \partial_l \xi^\psi_2)] + B \partial_l|\mathbf{e}_\perp|^2[B \partial_l \xi^\psi_2]
\]

\[+ \frac{\mathbf{e}_\perp \cdot (2 \gamma p \kappa - \rho \mathbf{g})}{1 + \gamma \beta} [\xi^\psi_2, B \partial_l \xi^\parallel_2] \]

\[+ \left[ \frac{\rho g \cdot \mathbf{B}}{B^2(1 + \gamma \beta)} \mathbf{e}_\perp \cdot (2B^2 \mathbf{\kappa} + \rho \mathbf{g}) - \mathbf{e}_\perp \cdot \mathbf{g} \mathbf{B} \cdot \nabla_0 \rho \right] [\xi^\psi_2, \xi^\parallel_2], \quad (21)\]

where the nonlinear convection term enters on the left side of the equation, the second and third terms on the right represent nonlinear field line bending, and the last two terms represent nonlinear compressional effects. The governing equations for the ballooning filament growth in the intermediate nonlinear regime \((\varepsilon_\xi \sim n^{-1/2})\) can be written in the following compact form

\[
\left[ \Psi + \xi^\psi_2, \rho |\mathbf{e}_\perp|^2 \partial^2 \xi^\parallel_2 - \mathcal{L}_\perp(\xi^\psi_2, \xi^\parallel_2) \right] = 0, \quad (22)
\]

\[
\rho B^2 \partial^2 \xi^\parallel_2 - \mathcal{L}_\parallel(\xi^\psi_2, \xi^\parallel_2) = 0. \quad (23)
\]

The above equations recover the nonlinear equations for the filament growth due to line-tied \(g\) mode in a shearless slab configuration [10, 11].

In general equations (22) and (23) require numerical solution. However, the structure of these two equations indicates that the solution satisfies the following two equations

\[
\rho |\mathbf{e}_\perp|^2 \partial^2 \xi^\psi_2 = \mathcal{L}_\perp(\xi^\psi_2, \xi^\parallel_2) + N(\Psi + \xi^\psi_2, l, t), \quad (24)
\]

\[
\rho B^2 \partial^2 \xi^\parallel_2 = \mathcal{L}_\parallel(\xi^\psi_2, \xi^\parallel_2) \quad (25)
\]
where \( N(\tilde{\Psi}, l, t) \) is a function of the distorted flux function \( \tilde{\Psi} = \Psi + \xi_1^\Psi \), in addition to the field line coordinate \( l \) and time. A particular choice is \( N(\tilde{\Psi}, l, t) = 0 \), which implies that the solutions of the linear local ballooning mode equations will continue to be the solution of the nonlinear ballooning equations (22) and (23). The nonlinearities in equations (22) and (23) would vanish for any solution that assumes the linear ballooning mode structure. As a consequence, the mode will grow exponentially at the growth rate of the corresponding linear phase even in the intermediate nonlinear stage. This may explain the depletion of nonlinearity previously conjectured for a nonlinear line-tied \( g \)-mode simulation [16], similar to that in an inviscid Euler flow [17].

III. BALLOONING FILAMENT EQUATIONS IN TOROIDAL FLUX COORDINATES

Next we consider the toroidal confinement systems in a magnetic flux coordinate system \((\Psi_0, \alpha_0, \Theta_0)\) so that \( \mathbf{B} = \nabla_0 \Psi_0 \times \nabla_0 \alpha_0 = B^\Theta \mathbf{e}_\Theta \), where \( \alpha_0 = q(\Psi_0) \Theta_0 - \zeta_0 \), \( q(\Psi_0) \) is the safety factor, \( \Theta_0 \) and \( \zeta_0 \) are poloidal and toroidal angle variables, respectively. Ignoring the \( g \)-mode, the nonlinear ballooning mode equations are

\[
\rho |e_\perp|^2 (\partial_{\alpha} \partial_2^2 \xi_2^\Psi + [\xi_2^\Psi, \partial_2^2 \xi_2^\Psi]) = \partial_\alpha \mathcal{L}_\perp (\xi_2^\Psi, \xi_2^\parallel) + |e_\perp|^2 [\xi_2^\Psi, B^\Theta \partial_\Theta (B^\Theta \partial_\Theta \xi_2^\Psi)] + B^\Theta \partial_\Theta |e_\perp|^2 [\xi_2^\Psi, B^\Theta \partial_\Theta \xi_2^\Psi] + \frac{2\gamma p e_\perp \cdot \kappa}{1 + \gamma \beta} [\xi_2^\Psi, B^\Theta \partial_\Theta \xi_2^\parallel],
\]

(26)

\[
\rho B^2 \partial_2^2 \xi_2^\parallel = \mathcal{L}_\parallel (\xi_2^\Psi, \xi_2^\parallel),
\]

(27)

where

\[
\mathcal{L}_\perp (\xi_2^\Psi, \xi_2^\parallel) = B^\Theta \partial_\Theta (|e_\perp|^2 B^\Theta \partial_\Theta \xi_2^\Psi) + 2e_\perp \cdot \kappa e_\perp \cdot \nabla_0 \kappa \xi_2^\Psi + \frac{2\gamma p e_\perp \cdot \kappa}{1 + \gamma \beta} \left( B^\Theta \partial_\Theta \xi_2^\parallel - 2e_\perp \cdot \kappa \xi_2^\Psi \right),
\]

(28)

\[
\mathcal{L}_\parallel (\xi_2^\Psi, \xi_2^\parallel) = B^\Theta \partial_\Theta \left[ \frac{\gamma p}{1 + \gamma \beta} \left( B^\Theta \partial_\Theta \xi_2^\parallel - 2e_\perp \cdot \kappa \xi_2^\Psi \right) \right].
\]

(29)

The differences here from previous section are the choice of \( \Theta_0 \) as the measure of the distance along the equilibrium magnetic field line, and the particular choice of \( \alpha_0 \) in terms of the poloidal and toroidal angle variables.
IV. LARGE ASPECT RATIO TOKAMAK IN SHAFRANOV COORDINATES

For tokamaks with large aspect ratio \(\epsilon = a/R_0 \ll 1\), where \(a\) is the minor radius, and \(R_0\) is the major radius), the configuration may be conveniently represented in the Shafranov coordinates \((r, \theta, \zeta)\) defined in the cylindrical coordinate system \((R, Z, \phi)\) as \(R = R_0 + r \cos \theta - \Delta(r), Z = r \sin \theta, \phi = -\zeta\). Here \(R_0\) is the major radius of magnetic axis, \(\Delta(r)\) is the Shafranov shift, and the subscripts “0” in the Lagrangian coordinates \((r_0, \theta_0, \zeta_0)\) are dropped for simplicity of notation. The toroidal flux coordinates \((\Psi_0, \alpha_0, \Theta_0)\) can be transformed to a field-line Shafranov coordinates \((r, \alpha, \theta)\) through asymptotic expansion in the inverse aspect ratio \(\epsilon\).

Specifically,
\[
\Psi_0 = \epsilon^{-1} \Psi(0)(r) + \epsilon \Psi(1)(r, \theta) + \cdots, \tag{30}
\]
\[
\Theta_0 = \Theta(0)(r, \theta) + \epsilon \Theta(1)(r, \theta) + \epsilon^2 \Theta(2)(r, \theta) + \cdots, \tag{31}
\]
\[
\alpha_0 = \alpha = q(\Psi_0(r, \theta))\Theta_0(r, \theta) - \zeta. \tag{32}
\]

From the tokamak magnetic configuration \(B = I(\Psi_0)\nabla \zeta_0 + \nabla \zeta_0 \times \nabla \Psi_0\) and the corresponding Grad-Shafranov equation \(R^2 \nabla \cdot (\nabla \Psi_0/R^2) + R^2 dp/d\Psi_0 + I dI/d\Psi_0 = 0\), we have
\[
\frac{[(r\Psi'(0))^2]}{2r^2 R_0^2} + p' + \frac{I'}{R_0^2} = 0, \tag{33}
\]
\[
r^{-1} \partial_r (r \partial_r \Psi(1)) + r^{-2} \partial^2_\theta \Psi(1) + \left[ \frac{d^2}{d\Psi_0^2} \left( \frac{R_0^2}{R_0^2} + \frac{I^2}{2} \right) \right] \Psi(1)
\]
\[
= - \left\{ 2r R_0 \frac{dp}{d\Psi_0} - \left[ \frac{1}{R_0} - \frac{(r \Delta')'}{r} \right] \Psi'(0) + 2 \Delta' \Psi''(0) \right\} \cos \theta \tag{34}
\]
where \((\cdot)' = d(\cdot)/dr_0\). Choosing \(\Psi(1) = 0\) and \(\Delta \neq 0\) allows \(r\) to remain a flux label for each circular flux surface with a shifted center, accurate to second order in \(\epsilon\). Using \(\Psi(0)(r) = rI/(qR_0)\), where \(I = I(r) + \mathcal{O}(\epsilon)\) and \(q = q(r) + \mathcal{O}(\epsilon^2)\), the first two leading order expansions of the Grad-Shafranov equation, (33) and (34), can be rewritten as follows [18, 19]:
\[
\mathcal{O}(\epsilon^0) : \quad \frac{I^2}{q R_0} \left( \frac{r^2}{q} \right)' + p' + \frac{I'}{R_0^2} = 0, \tag{35}
\]
\[
\mathcal{O}(\epsilon) : \quad \epsilon \Delta'' + (3 - 2s) \Delta' - \frac{r}{R_0} + 2 \left( \frac{q R_0}{I} \right)^2 R_0 p' = 0. \tag{36}
\]
Here \(s = rq'/q\) defines the magnetic shear. The orderings \(R_0 \sim \epsilon^{-1}, \Delta \sim \Delta' \sim \Delta'' \sim \epsilon, I \sim \epsilon^{-2}, I' \sim \epsilon^0\), and \(p \sim p' \sim \epsilon^0\) are assumed for a low \(\beta\) tokamak in large aspect ratio limit in obtaining (33) to (36). Similarly, for the poloidal angle variable we have
\[
\partial_\theta \Theta_0 = 1 - \epsilon \left( \frac{r}{R_0} + \Delta' \right) \cos \theta + \mathcal{O}(\epsilon^2) \tag{37}
\]
so that Θ(0) = θ - θ0, and Θ(1) = -(r/R0 + Δ')(sin θ - sin θ0), where θ0 has the conventional dual role in ballooning theory as an integration constant and a normalized radial wavenumber [6].

With the above transformation from (30) to (32), we have \( \mathbf{B} \cdot \nabla = B^\theta \partial_\theta = [1 + \mathcal{O}(\epsilon^2)]B^\theta \partial_\theta \), where
\[
B^\theta = \frac{I}{qR_0^2} \left[ 1 - \epsilon \left( \frac{r}{R_0} - \Delta' \right) \cos \theta + \mathcal{O}(\epsilon^2) \right].
\]
The nonlinear ballooning equations in Eqs. (28) and (29) take the following forms in Shafranov coordinates, valid to second order in \( \epsilon \)

\[
\rho h(\theta)(\partial_\alpha \partial^2_\beta \xi^\alpha_2 + [\xi^\alpha_2, \partial^2_\beta \xi^\alpha_2]) = \partial_\alpha \mathcal{L}_\perp(\xi^\alpha_2, \xi^\parallel_2)
\]
\[
+ \partial_\alpha h(\theta)[\xi^\alpha_2, B^\theta \partial_\theta (B^\theta \partial_\theta \xi^\alpha_2)] + B^\theta \partial_\theta h(\theta)[\xi^\alpha_2, B^\theta \partial_\theta \xi^\alpha_2] + 2\gamma p g(\theta) \left[ \frac{1 + \gamma \beta}{1 + \gamma \beta} \left[ \xi^{\parallel_2, \parallel_2} - 2g(\theta)\xi^{\parallel_2, \parallel_2} \right] \right].
\]

\[
\rho B^2 \partial^2_\alpha \xi^\parallel_2 = \mathcal{L}_\parallel(\xi^\alpha_2, \xi^\parallel_2)
\]

where \( \xi^\parallel_2 = \xi^\parallel_2 \cdot \nabla r \), and

\[
[A, B] = [A, B]_r + \epsilon \left( \frac{1}{R_0} + \Delta'' \right) [A, B]_\theta,
\]

\[
[A, B]_r = \partial_\alpha A \partial_\beta B - \partial_\alpha A \partial_\beta B,
\]

\[
[A, B]_\theta = \partial_\theta A \partial_\alpha B - \partial_\alpha A \partial_\theta B.
\]

The linear ballooning operators become two ordinary differential operators along \( \theta \)

\[
\mathcal{L}_\perp(\xi^\alpha_2, \xi^\parallel_2) = B^\theta \partial_\theta (h(\theta)B^\theta \partial_\theta \xi^{\parallel_2}) + 2g(\theta)p'\xi^{\parallel_2}
\]
\[
+ \frac{2\gamma p g(\theta)}{1 + \gamma \beta} \left( B^\theta \partial_\theta \xi^{\parallel_2} - 2g(\theta)\xi^{\parallel_2} \right),
\]

\[
\mathcal{L}_\parallel(\xi^\alpha_2, \xi^\parallel_2) = B^\theta \partial_\theta \left[ \frac{\gamma p}{1 + \gamma \beta} \left( B^\theta \partial_\theta \xi^{\parallel_2} - 2g(\theta)\xi^{\parallel_2} \right) \right]
\]

and the major ballooning geometric factors in Shafranov coordinates are [4, 19]

\[
g(\theta) = \Psi'_0 e_{\parallel} \cdot \kappa
\]
\[
= -\frac{\epsilon}{R_0} \left[ \cos \theta + s(\theta - \theta_0) \sin \theta \right]
\]
\[
- \frac{\epsilon^2 r}{R_0^2} \left[ \frac{1}{q'^2} - \frac{r_0 \Delta'}{R_0} \left[ 1 - \frac{s(\theta - \theta_0) \sin 2\theta}{2} \right] - \cos^2 \theta \frac{M'}{q} \left( \sin \theta - \sin \theta_0 \right) \sin \theta \right] + \mathcal{O}(\epsilon^3)
\]
\[
= -\frac{\epsilon}{R_0} \left[ R + \cos \theta + s(\theta - \theta_0) - \epsilon \alpha_x (\sin \theta - \sin \theta_0) \right] \sin \theta \right] + \mathcal{O}(\epsilon^2),
\]

\[
h(\theta) = |\Psi'_0 e_{\perp}|^2
\]
\[ = 1 + s^2(\theta - \theta_0)^2 - \epsilon \left\{ \frac{2}{q} \left( \frac{rM' + \Delta'}{qR_0} \right) s(\theta - \theta_0)(\sin \theta - \sin \theta_0) \right. \\
\left. + 2\Delta's(\theta - \theta_0)\sin \theta_0 + 2 \left[ \frac{M'}{qR_0} s^2(\theta - \theta_0)^2 \cos \theta \right] \right\} + \mathcal{O}(\epsilon^2) \]
\[ = 1 + [s(\theta - \theta_0) - \epsilon \alpha_p(\sin \theta - \sin \theta_0)]^2 + \mathcal{O}(\epsilon) \]  

where \[ M = q(r + R_0\Delta'), \quad \alpha_p = \frac{rM'}{qR_0} + \Delta', \quad \bar{\kappa} = \frac{\epsilon r}{R_0} \left( \frac{1}{q^2} - \frac{R_0\Delta'}{r} \right). \]  

Note that the spatial derivatives of the plasma displacement \( \xi \) are operated on the normalized coordinate \((r, \alpha, \theta) = (\sqrt{nr_0}, n\alpha_0, \theta_0)\), whereas the spatial derivatives of the plasma equilibrium are operated on the coordinate \((r_0, \alpha_0, \theta_0)\) directly. The parameter \( \alpha_p \) is related to the equilibrium pressure gradient through the Grad-Shafranov equation in (35) and (36), and is given by \[ \alpha_p = -2 \left( \frac{qR_0}{I} \right)^2 R_0p' + \frac{r}{R_0}(2 + s) - \Delta'(2 - 3s) \]  

which reduces to \( \alpha_p \approx -2(qR_0/I)^2 R_0p' \) when \( p' \sim \epsilon^{-1} \) near ballooning stability threshold [4, 19]. At leading order in \( \epsilon \), it can be seen that the geometric factors \( g(\theta) \) and \( h(\theta) \) assume the form of the \( s - \alpha \) model, which is commonly used in the study of linear ballooning instability [4]. We have thus obtained the ballooning filament growth equations for the intermediate nonlinear regime in the \( s - \alpha \) model as well.

V. SUMMARY

An ideal MHD theory for the filament growth due to a ballooning instability in the intermediate nonlinear regime has been developed in general toroidal magnetic configurations. The nonlinear equations are further explicitly formulated in a Shafranov coordinate for a circular-shaped tokamak plasma in the limit of large aspect ratio. The equations constitute an \( s - \alpha \) model for the intermediate nonlinear ballooning instability.

In this nonlinear regime, there are three major nonlinear effects that are involved in the development of a ballooning filament. From the nonlinear equation for \( \xi_\Psi \) in Eq. (26) or Eq. (39), those nonlinear effects are due to radial convection, line bending, and magnetosonic coupling. The physical consequences of each of those nonlinear effects are yet to be explored.

A remarkable feature of the nonlinear equations is that solutions of the associated local linear ballooning mode equations continue to be valid solutions into the intermediate non-
linear regime. This observation may be consistent with previous numerical simulations of the line-tied $g$ mode which obtained similar results [11, 16].

In next step we plan to solve the nonlinear $s - \alpha$ model in Eqs. (39) and (40) both analytically and numerically. We then will compare the solutions with results from direct MHD simulations, using the NIMROD code [20], of nonlinear ballooning instability in a circular tokamak with large aspect ratio. The goal is to determine the nonlinear ballooning filament growth and saturation properties and how they depend upon the structure of the pedestal region. This will be a step toward understanding the precursor and onset phases of ELMs.
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