Filamentary Structure and Exponential Growth of Nonlinear Ballooning Instability

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A ballooning instability in a tokamak (NIMROD simulation)
Filamentary structure persists during (type-I) edge localized modes (ELMs) in the MAST tokamak [Kirk et al., 2006].

Both linear and non-linear ELM phases.
ELM filaments resemble linear ballooning mode structure

- Characteristics of ELM filaments
  - Toroidal mode number $n \sim 15 \text{–} 20$
  - Elongated structures aligned with field lines
  - Persist well into nonlinear phase

- Questions to address in theory:
  - Why do nonlinear ELM and ballooning filaments resemble linear ballooning structure?
  - What is the temporal evolution in the nonlinear regime?
Different nonlinear regimes of ballooning instability are characterized by the relative strength of the nonlinearity with powers of $n^{-1}$

- Nonlinearity and ballooning parameters
  \[
  \varepsilon \sim \frac{|\xi|}{L_{\text{eq}}} \ll 1, \quad n^{-1} \sim \frac{k_{||}}{k_{\perp}} \sim \frac{L_y}{L_z} \ll 1
  \]

- For $\varepsilon \ll n^{-1}$, linear ballooning mode theory [Coppi, 1977; Connor, Hastie, and Taylor, 1979; Dewar and Glasser, 1983]

- For $\varepsilon \sim n^{-1}$, early nonlinear regime [Cowley and Artun, 1997; Hurricane, Fong, and Cowley, 1997; Wilson and Cowley, 2004]

- For $\varepsilon \sim n^{-1/2}$, intermediate nonlinear regime → this talk [Zhu, Hegna, and Sovinec, 2006; Zhu et al., 2007; Zhu and Hegna, 2008; Zhu, Hegna, and Sovinec, 2008; ]

- For $\varepsilon \gg n^{-1/2}$, late nonlinear regime; analytic theory under development.
Intermediate nonlinear regime was previously identified for line-tied $g$ mode

Left: Equilibrium configuration; Right: $u_x$ contour of line-tied $g$-mode

\[ \frac{d}{dx_0} \left( \rho_0 + \frac{B_0^2}{2} \right) = \rho_0 \mathbf{g} \cdot \hat{x}, \quad \mathbf{g} = -g \hat{x}, \quad \mathbf{B}_0 = B_0 \hat{z} \]

\[ \rho_0(x_0) = \rho_c + \rho_h \tanh \left( x_0 + L_c \right)/L_\rho, \quad \rho_0(x_0) = \rho_0(x_0) \]

$L_\rho \rightarrow$ pedestal width; $2 \rho_h \rightarrow$ pedestal height
Transition from early nonlinear regime $\varepsilon \sim n^{-1}$ to intermediate nonlinear regime $\varepsilon \sim n^{-1/2}$ [Zhu et al., 2007]

$$(u_x)_{\text{max}} (t = 0) = 10^{-3}$$

$t_1 \sim 46$: $\varepsilon \sim n^{-1}$, $u_x \sim 0.008$, $\xi_x \sim 0.1 \rightarrow$ mode width in $y$;

$t_2 \sim 80$: $\varepsilon \sim n^{-1/2}$, $u_x \sim 0.3$, $\xi_x \sim 3.6 \rightarrow$ mode width in $x$.

$t_1 \lesssim t \lesssim t_2$: finger formation initiates during the transition;

$t_2 \lesssim t \lesssim t_3$: finger pattern becomes prominent as the mode proceeds through the regime $\varepsilon \sim n^{-1/2}$. 
Exponential growth persists in intermediate nonlinear regime of the line-tied $g$-mode [Zhu et al., 2007]

$$(u_x)_{\text{max}}(t = 0) = 10^{-5}$$

Simulation results agree with numerical solution of the nonlinear line-tied $g$ mode equations.

Both simulation and numerical solution of theory indicate exponential-like nonlinear growth. Why?
1. Nonlinear ballooning equations
   ▶ Formulation
   ▶ Analytic solution
2. Comparison with NIMROD simulations
   ▶ Simulation setup
   ▶ Comparison method
   ▶ Comparison results
3. Summary and Discussion
A Lagrangian form of ideal MHD is used to develop the theory of nonlinear ballooning instability

$$\frac{\rho_0}{J} \nabla_0 r \cdot \frac{\partial^2 \xi}{\partial t^2} = -\nabla_0 \left[ \frac{\rho_0}{J^\gamma} + \frac{(B_0 \cdot \nabla_0 r)^2}{2J^2} \right]$$

$$+ \nabla_0 r \cdot \left[ \frac{B_0}{J} \cdot \nabla_0 \left( \frac{B_0}{J} \cdot \nabla_0 r \right) \right] + \frac{\rho_0}{J} \nabla_0 r \cdot \mathbf{g}$$  \hspace{1cm} (1)

where \( r(r_0, t) = r_0 + \xi(r_0, t), \nabla_0 = \frac{\partial}{\partial r_0}, J(r_0, t) = |\nabla_0 r| \hspace{1cm} (2) \)

The full MHD equation can be further reduced for nonlinear ballooning instability using expansion in terms of \( \varepsilon \) and \( n^{-1} \)

$$\xi(\sqrt{n}x_0, ny_0, z_0, t) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \varepsilon^i n^{-\frac{i}{2}} \left( \hat{x} \xi_x\{i,j\} + \frac{\hat{y}}{\sqrt{n}} \xi_y\{i,j\} + \hat{z} \xi_z\{i,j\} \right) \hspace{1cm} (3)$$
Nonlinear ballooning expansion is carried out for general magnetic configurations with flux surfaces

- Clebsch coordinate system \((\psi_0, \alpha_0, l_0)\)

\[
B = \nabla_0 \psi_0 \times \nabla_0 \alpha_0 \tag{4}
\]

- Expansions are based on intermediate nonlinear ballooning ordering \(\varepsilon \equiv |\xi|/L_{\text{eq}} \sim n^{-1/2} \ll 1\)

\[
\xi(\sqrt{n}\psi_0, n\alpha_0, l_0, t) = \sum_{j=1}^{\infty} n^{-\frac{j}{2}} \left( e_\perp \xi_j^{\psi} + \frac{e^\wedge}{\sqrt{n}} \xi_{j+1}^{\alpha} + B\xi_j^\parallel \right) \tag{5}
\]

\[
J(\sqrt{n}\psi_0, n\alpha_0, l_0, t) = 1 + \sum_{j=0}^{\infty} n^{-\frac{j}{2}} J_j^{\frac{1}{2}} \tag{6}
\]

where \(e_\perp = (\nabla_0 \alpha_0 \times B)/B^2, e^\wedge = (B \times \nabla_0 \psi_0)/B^2\).

- The spatial structure of \(\xi(\psi, \alpha, l)\) and \(J(\psi, \alpha, l)\) is ordered to be consistent with linear ideal ballooning theory:
  \(\psi = \sqrt{n}\psi_0, \alpha = n\alpha_0, l = l_0\).
The linear local ballooning operator will continue to play a fundamental role in the nonlinear dynamics

Linear ideal MHD ballooning and $g$ modes obey

$$\rho \partial_t^2 \xi = \mathcal{L}(\xi)$$  \hspace{1cm} (7)

where

$$\mathcal{L}(\xi) \equiv B \cdot \nabla_0 (B \cdot \nabla_0 \xi) - \left[ \nabla_0 (B \cdot \nabla_0 B) + \nabla_0 (\rho g) \right] \cdot \xi$$

$$- \frac{BB \cdot \nabla_0 }{1 + \gamma \beta} \left\{ \frac{1}{B^2} \left[ B \cdot \nabla_0 \xi \| - \frac{\rho B \cdot g}{B^2} \xi \| - \left( 2\kappa + \frac{\rho g}{B^2} \right) \cdot \xi \| \right] \right\}$$

$$- \frac{2B \cdot \nabla_0 B + \rho g}{1 + \gamma \beta} \left[ B \cdot \nabla_0 \xi \| - \frac{\rho B \cdot g}{B^2} \xi \| - \left( 2\kappa + \frac{\rho g}{B^2} \right) \cdot \xi \| \right]$$  \hspace{1cm} (8)

is an ODE operator, with $\xi = \xi \| B + \xi \| \perp$ [Hurricane, Fong, and Cowley, 1997].
A set of nonlinear ballooning equations for $\xi$ are described using the linear operator [Zhu and Hegna, 2008]

\[
\rho(\|e\|_\perp^2 \partial_\alpha \partial_t^2 \xi_{1/2}^\Psi + [\xi_{1/2}^1, \partial_t^2 \xi_{1/2}^1]) = \partial_\alpha \mathcal{L}_{\perp}(\xi_{1/2}^\Psi, \xi_{1/2}^\parallel) + [\xi_{1/2}^1, \mathcal{L}(\xi_{1/2}^1)],
\]

(9)

\[
\rho B^2 \partial_t^2 \xi_{1/2}^\parallel = \mathcal{L}_{\parallel}(\xi_{1/2}^\Psi, \xi_{1/2}^\parallel)
\]

(10)

\[
\mathcal{L}_{\perp}(\xi^\Psi, \xi^\parallel) \equiv e_{\perp} \cdot \mathcal{L}(\xi)
\]

(11)

\[
= B \partial_l(\|e\|_\perp^2 B \partial_l \xi^\Psi) + 2e_{\perp} \cdot \kappa e_{\perp} \cdot \nabla_0 \rho \xi^\Psi
\]

\[
\quad + \frac{2\gamma \rho e_{\perp} \cdot \kappa}{1 + \gamma \beta} \left( B \partial_l \xi^\parallel - 2e_{\perp} \cdot \kappa \xi^\Psi \right),
\]

(12)

\[
\mathcal{L}_{\parallel}(\xi^\Psi, \xi^\parallel) \equiv B \cdot \mathcal{L}(\xi)
\]

(13)

\[
= B \partial_l \left[ \frac{\gamma \rho}{1 + \gamma \beta} \left( B \partial_l \xi^\parallel - 2e_{\perp} \cdot \kappa \xi^\Psi \right) \right],
\]

(14)

\[
[A, B] \equiv \partial_\psi A \cdot \partial_\alpha B - \partial_\alpha A \cdot \partial_\psi B.
\]

(15)
The local linear ballooning mode structure and growth continue to satisfy the nonlinear ballooning equations in Lagrangian space

- The nonlinear ballooning equations can be written in the compact form

\[
\begin{bmatrix}
\Psi + \xi\Psi, \rho|e_\perp|^2 \partial_t^2 \xi\Psi - \mathcal{L}_\perp(\xi\Psi, \xi\parallel) \\
\rho B^2 \partial_t^2 \xi\parallel - \mathcal{L}_\parallel(\xi\Psi, \xi\parallel)
\end{bmatrix} = 0, \quad (16)
\]

\[
\rho|e_\perp|^2 \partial_t^2 \xi\Psi = \mathcal{L}_\perp(\xi\Psi, \xi\parallel) + N(\Psi + \xi\Psi, l, t), \quad (18)
\]

\[
\rho B^2 \partial_t^2 \xi\parallel = \mathcal{L}_\parallel(\xi\Psi, \xi\parallel). \quad (19)
\]

- The general solution satisfies

\[
\begin{align*}
\rho|e_\perp|^2 \partial_t^2 \xi\Psi &= \mathcal{L}_\perp(\xi\Psi, \xi\parallel) + N(\Psi + \xi\Psi, l, t), \\
\rho B^2 \partial_t^2 \xi\parallel &= \mathcal{L}_\parallel(\xi\Psi, \xi\parallel).
\end{align*}
\]

- A special solution to the nonlinear ballooning equations is the solution of the linear ballooning equations

\[
N(\tilde{\Psi}, l, t) = 0, \quad \text{where} \quad \tilde{\Psi} = \Psi + \xi\Psi. \quad (20)
\]
Implications of the analytic solution

- The solution is linear in Lagrangian coordinates, but nonlinear in Eulerian coordinates

\[ \xi = \xi_{\text{lin}}(r_0) = \xi_{\text{lin}}(r - \xi) = \xi_{\text{non}}(r). \]

- Linear ballooning mode operator requires solution having filamentary structure \( \xi \sim A(\psi, \alpha)H(l) \)

- Linear ballooning mode structure gives exponential growth

\[ \partial_t^2 \xi = \mathcal{L} \xi = \Gamma^2 A(\psi, \alpha)H(l) = \Gamma^2 \xi. \]

- Perturbation developed from linear ballooning instability should continue to
  - grow exponentially
  - maintain filamentary spatial structure
Outline

1. Nonlinear ballooning equations
   - Formulation
   - Analytic solution

2. Comparison with NIMROD simulations
   - Simulation setup
   - Comparison method
   - Comparison results

3. Summary and Discussion
Simulations of ballooning instability are performed in a tokamak equilibrium with circular boundary and pedestal-like pressure.

- Equilibrium from ESC solver [Zakharov and Pletzer, 1999]
- Finite element mesh in NIMROD simulation.
Linear ballooning dispersion is characteristic of interchange type of instabilities

![Graph showing linear ballooning dispersion](f_050808x01x02)

Extensive benchmarks between NIMROD and ELITE show good agreement [B. Squires et al., 2009].
Simulation starts with a single $n = 15$ linear ballooning mode.
Isosurfaces of perturbed pressure $\delta p$ show filamentary structure ($t = 30\mu s$, $\delta p = 168\, Pa$)
For theory comparison, we need to know plasma displacement $\xi$ associated with nonlinear ballooning instability

$\xi$ connects the Lagrangian and Eulerian frames,

$$r(r_0, t) = r_0 + \xi(r_0, t)$$

(21)

In the Lagrangian frame

$$\frac{d\xi(r_0, t)}{dt} = u(r_0, t)$$

(22)

In the Eulerian frame

$$\partial_t \xi(r, t) + u(r, t) \cdot \nabla \xi(r, t) = u(r, t)$$

(23)

where $u(r, t)$ is velocity field, $\partial_t = (\partial/\partial t)_r$, and $\nabla = \partial/\partial r$. $\xi$ is advanced as an extra field in simulations.
Lagrangian compression $\nabla_0 \cdot \xi$ can be more conveniently used to identify nonlinear regimes

- Nonlinearity is defined by $|\xi|/L_{eq}$, but $L_{eq}$ is not specific.
- Linear regime
  \[ \nabla_0 \cdot \xi = \nabla \cdot \xi \] (24)
- Early nonlinear regime
  \[ \nabla_0 \cdot \xi \sim \lambda^{-1}_\psi \xi^\psi + \lambda^{-1}_\alpha \xi^\alpha + \lambda^{-1}_\parallel \xi^\parallel \] (25)
  \[ \sim n^{1/2} n^{-1} + n^1 n^{-3/2} + n^0 n^{-1} \sim n^{-1/2} \ll 1. \]
- Intermediate nonlinear regime
  \[ \nabla_0 \cdot \xi \sim \lambda^{-1}_\psi \xi^\psi + \lambda^{-1}_\alpha \xi^\alpha + \lambda^{-1}_\parallel \xi^\parallel \sim n^{1/2} n^{-1/2} + n^1 n^{-1} + n^0 n^{-1/2} \sim 1. \] (26)
- The Lagrangian compression is sensitive to nonlinearity: matrix $(I - \nabla \xi)^{-1}$ could become singular passing beyond intermediate nonlinear regime.
The Lagrangian compression \( \nabla_0 \cdot \xi \) is calculated using the Eulerian tensor \( \nabla \xi \)

Transforming from Lagrangian to Eulerian frames, one finds

\[
\xi(r_0, t) = \xi[r - \xi(r, t), t] \quad (27)
\]

\[
\nabla \xi = \frac{\partial \xi}{\partial r}
\]

\[
= \left( \frac{\partial r}{\partial r} - \frac{\partial \xi}{\partial r} \right) \cdot \frac{\partial \xi}{\partial r_0}
\]

\[
= (I - \nabla \xi) \cdot \nabla_0 \xi \quad (28)
\]

The Lagrangian compression \( \nabla_0 \cdot \xi \) is calculated from the Eulerian tensor \( \nabla \xi \) at each time step using

\[
\nabla_0 \cdot \xi = \text{Tr}(\nabla_0 \xi) = \text{Tr}[ (I - \nabla \xi)^{-1} \cdot \nabla \xi ] . \quad (29)
\]
Exponential linear growth persists in the intermediate nonlinear regime of tokamak ballooning instability [Zhu, Hegna, and Sovinec, 2008]

Dotted line indicates the transition to the intermediate nonlinear regime when $\nabla_0 \cdot \xi \sim O(1)$
Lagrangian compression may also be able to identify other nonlinear ballooning regimes.

- Intermediate nonlinear regime is entered $\sim 28\mu s$.
- Large $\nabla_0 \cdot \xi$ indicates transition to nonlinear regimes.
Perturbation energy grows with the linear growth rate into the intermediate nonlinear regime (vertical line)

![Graph showing perturbation energy growth over time](image_url)
Contours of plasma velocity and displacement $\xi$ at $t = 5\mu s$, linear phase
Contours of plasma velocity and displacement $\xi$ at $t = 30\mu s$, intermediate nonlinear phase
Contours of plasma velocity and displacement $\xi$ at $t = 40\mu s$, start of late nonlinear phase
Distortions of flux surface are consistent with plasma displacement

Above: Total pressure contours.
Left: $t = 30\mu s$; Right: $t = 40\mu s$. 
Total pressure contours on 4 poloidal slices show 3D view of distorted pedestal ($t = 40 \mu s$)
Total pressure isosurface shows nonlinear filamentary structure \((t = 40\mu s, p = 3602\text{Pa})\)
Distribution of plasma displacement vectors ($\xi$) aligns with pressure isosurface ribbons ($t = 40 \mu s$)
Zoom-in view of total pressure isosurface and displacement ($\xi$) vectors ($t = 40\mu s$)
Distribution of velocity \((u)\) vector is different from that of displacement \((\xi)\) vector at \(t = 40 \mu s\)
Simulations started from multiple linear ballooning modes also confirm theory prediction.
Summary

- There is an exponential growth phase of nonlinear ballooning instability
  - The growth rate is same as the linear growth rate.
  - The spatial structure is same as the linear mode (in Lagrangian space).
  - This nonlinear phase is defined by ordering $\xi_\psi \sim \lambda_\psi$.

- Implications
  - May explain persistence of filamentary structures in ELM experiments.
  - Nonlinear ELM behavior of pedestal may be largely controlled by its linear properties (which is determined by the configuration).
Discussion

- What are missing?
- Nonlinearity
  - What is the “nonlinearity depletion” mechanism?
  - Marginal unstable configuration ($\Gamma \sim 0$) need nonlinear filament envelope equation (detonation?)
  - Late nonlinear regimes: saturation, filament->blob?
- 2-fluid, FLR (finite Larmor radius) effects
- Edge shear flow effects
- RMP (resonant magnetic perturbation) effects
- Geometry (non-circular shape, divertor separatrix/X-point)
- Peeling-ballooning coupling
- ......