Stability and Robustness of Wholesale Electricity Markets ⋆

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Abstract: We present stability and robustness conditions of wholesale electricity markets. The analysis makes use of a control-theoretic market analysis framework that merges concepts of market efficiency, Lyapunov stability, game-theory, and predictive control.

Keywords: electricity, markets, stability, robustness, game-theory.

1. INTRODUCTION

Understanding the sources of instability of electricity markets has significant economic implications. In particular, market instability leads to strong price fluctuations and to inefficient spread of social welfare among consumers and producers. Most market studies have used game-theoretical models to understand *static* stability properties. While these studies provide some insights into the major factors affecting market performance, they cannot explain the emergence of long-term instability arising from effects propagated forward in time.Examples of these factors are ramp constraints, renewable supply, and suboptimal gaming solutions.de la Torre et al. (2003); Mookherjee et al. (2008) and Kannan and Zavala (2010) have used dynamic game-theoretical models to understand the effect of ramping constraints and renewable generation on long-term performance.

A caveat of market studies reported to date has been the lack of a coherent framework that enables a systematic assessment of the stabilizing properties of different market designs. Zavala and Anitescu (2011) constructed a control-theoretical framework using market efficiency, Lyapunov stability, and predictive control concepts. A market-specific Lyapunov function was derived by using a summarizing state that measures the progress of the market efficiency. This Lyapunov function was used to establish conditions under which a given market design can guarantee long-term stability. In particular, the framework can be used to explain how incomplete gaming solutions, short foresight horizons, and limited ramping capacity can lead to instability. These insights were used to propose a new stabilizing market design by making use of a stabilizing constraint for the market efficiency.

In this work, we extend the control-theoretic framework to analyze robustness properties of market designs. The paper is structured as follows. In Section 2 we present the market structure under consideration. In Section 3 we discuss implementation issues arising from incomplete gaming. In Section 4 we study nominal market stability properties. In Section 5 we extend the results to robust stability. In Section 6 we provide concluding remarks and recommendations for future extensions.

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2. MARKET STRUCTURE

We first define the market structure under consideration and discuss the underlying modeling assumptions.

2.1 Suppliers

There exist a myriad of electricity market models capturing implementation details in different parts of the world. We consider a supply-function equilibrium market structure similar to those proposed by Hui et al. (2005)and Kannan and Zavala (2010). This model reflects some basic aspects of actual implementations in US real-time markets. Here, the supplier decisions are the parameters a_k^i, b_k^i of the affine supply function:

$$q_k^i(p_k, b_k^i, a_k^i) = b_k^i \cdot (p_k - a_k^i). \tag{1}$$

Here, q_k^i is the production quantity of supplier $i \in \mathcal{S} := \{1..S\}$ at time k; $p_k \geq 0$ is the price at time k; and a_k^i, b_k^i are the bidding coefficients at time k for supplier i. We assume that the supply function is nondecreasing in p_k. Consequently, we impose the requirement that $b_k^i \geq 0$. In our analysis, we assume that the generation quantities q_k^i and p_k are always non-negative. Consequently, we restrict the intercept parameter a_k^i to be non-negative as well. The supply function can also be expressed in inverse form as

$$p_k(q_k^i, b_k^i, a_k^i) = \frac{1}{b_k^i} q_k^i + a_k^i. \tag{2}$$

We observe that multiple combinations of $a_k^i, b_k^i \geq 0$ can reach the same quantities or prices. Since this ill-posedness introduces difficulties in analyzing the properties of the supplier problem, we will assume that the intercept parameters a_k^i are zero. This assumption will not affect the analysis as long as the price is assumed to be non-negative. The consumer demands will be assumed to be fixed (inelastic) and denoted by d_k^j, where $j \in \mathcal{C} := \{1..C\}$ is the set of consumers.

The supplier problem can be posed as follows. Starting at time k, given the price signals p_k over the future horizon $\mathcal{T}_k := \{k..k+T\}$, where T is the horizon length and the current states q_k^i, b_k^i, find the bidding parameter trajectories $b_t^i,\ t \in \mathcal{T}_k$, that maximize the future profit (revenue minus marginal cost). We can pose the supplier problem entirely in terms of the prices p_k and the supply function parameters b_k^i. In addition, we can interpret the bidding parameters b_k^i as the suppliers states. These modifications lead to the following equivalent formulation in state-space form:

$$\max_{b_t^i, \Delta b_t^i} \sum_{t \in \mathcal{T}_k} \phi_t^i := \sum_{t \in \mathcal{T}_k} \left(p_t \cdot b_t^i \cdot p_t - c_t^i \left(b_t^i \cdot p_t \right) \right) \tag{3a}$$

$$\text{s.t. } b_{t+1}^i = b_t^i + \Delta b_t^i,\ t \in \mathcal{T}_k^{-} \tag{3b}$$

$$\underline{q}^i \leq b_t^i \cdot p_t \leq \overline{q}^i,\ t \in \mathcal{T}_k \tag{3c}$$

$$b_t^i \geq 0,\ t \in \mathcal{T}_k \tag{3d}$$

$$b_k^i = \text{given}, \tag{3e}$$

where $\underline{q}^i, \overline{q}^i \geq 0$ are the lower and upper generation limits, respectively. We also have $\mathcal{T}_k^{-} := \mathcal{T}_k \backslash \{k+T\}$. The bidding increments Δb_k^i are interpreted as the control actions of the supplier. Note that these are unconstrained, implying that the suppliers can adjust their bids infinitely fast. A direct consequence is that the feasible set of the problem is invariant to the initial states b_k^i. In addition, the feasible set is invariant to the price signals p_k since it is always possible to find $b_k^i \geq 0$ mapping any p_k to a feasible quantity q_k^i. Consequently, we denote the feasible set of this problem as Ω^i.

The accumulated future profit is denoted by $\sum_{t \in \mathcal{T}_k} \phi_t^i$. The marginal cost function is assumed to have the form

$$c_k^i(q_k^i) = h_k^i \cdot q_k^i + \frac{1}{2} g_k^i \cdot (q_k^i)^2. \tag{4}$$

We make the common assumption that $g_k^i > 0$ so the marginal cost is convex in q_k^i (see Rudkevich (2005)).

Property 2.1. If $p_t \geq 0$ and $g_t^i \geq 0,\ t \in \mathcal{T}_k$, then problem (3) is convex. If $p_t > 0$, the problem has a feasible solution for any $\underline{q}^i, \overline{q}^i \geq 0$. If $p_t = 0$, the problem admits a solution only if $\underline{q}^i = 0$.

2.2 ISO Market Clearing

The independent system operator (ISO) receives the bidding states b_k^i and clears the market by determining the generation quantities (and implicitly the prices) that balance total supply and demand. The main objectives of the ISO are to maximize social welfare and efficiency and to ensure market stability. The interaction between the ISO and the suppliers results in a game in which each player tries to maximize its own performance metric.

In our analysis, market stability will be interpreted as the ability to keep prices bounded from a given reference in the presence of dynamic fluctuations of demands and renewable supply and physical constraints. To account for this, we propose to use the basic concept of *market efficiency* as a measure of stability. To define efficiency, we first define an *ideal unconstrained* market clearing problem. This problem can be stated as follows. Given supply function states b_k^i, solve the following problem (see Baldick and Hogan (2002)):

$$\min_{q_t^i} \quad \sum_{t \in \mathcal{T}_k} \bar{\varphi}_t := \sum_{t \in \mathcal{T}_k} \sum_{i \in \mathcal{S}} \int_0^{q_t^i} p_t(q, b_t^i) dq \tag{5a}$$

s.t.

$$\sum_{i \in \mathcal{S}} q_t^i \geq \sum_{j \in \mathcal{C}} d_t^j,\ t \in \mathcal{T}_k \tag{5b}$$

$$\underline{q}^i \leq q_t^i \leq \overline{q}^i,\ i \in \mathcal{S}, t \in \mathcal{T}_k, \tag{5c}$$

where

$$\int_0^{q_k^i} p_k(q, b_k^i) dq = \frac{1}{2\, b_k^i} (q_k^i)^2. \tag{6}$$

The objective function is the *negative social welfare*, denoted as $\sum_{t \in \mathcal{T}_k} \bar{\varphi}_t$. Since we have assumed that the consumers do not bid into the market, the objective function reduces to the aggregated income of the suppliers. We have that $\bar{\varphi}_t \geq 0$ since $q_t^i, b_t^i \geq 0,\ t \in \mathcal{T}_k$. The multipliers for the constraint (5b), the demand satisfaction constraint, are the prices $\bar{p}_t \geq 0$. Note that the feasible set of this problem is not affected by the bidding parameters, since they enter only in the objective function. In addition, in this unconstrained formulation, we assume that the generators can move infinitely fast between production levels (no ramp constraints). This assumption decouples the problem in time (i.e., each time step can be solved separately). Hence, the feasible set of this problem is invariant to the current state of the generators q_k^i. Nevertheless, the feasible set of this problem does depend on the demands. Accordingly, the feasible set of this problem will be denoted as $\Omega_{UNC}^{ISO}(d_{\mathcal{T}_k}^j)$, where $d_{\mathcal{T}_k}^j := \{d_k^j, ..., d_{k+T}^j\}$.

Property 2.2. If $b_t^i \geq 0, t \in \mathcal{T}_k$, problem (5) is convex. The problem has a feasible solution if $\sum_{i \in \mathcal{S}} \underline{q}^i \leq \sum_{j \in \mathcal{C}} d_j^t \leq \sum_{i \in \mathcal{S}} \overline{q}^i$ holds. If $b_t^i > 0$, feasibility holds for any $\underline{q}^i, \overline{q}^i \geq 0$. If $b_t^i = 0$, the problem admits a solution only if $\underline{q}^i = 0$.

For our analysis, we note that having infinitely fast dynamics in the generators is equivalent to assuming that their ramp capacities are equal to the distance between the maximum and minimum generation capacities $\overline{q}^i - \underline{q}^i$. Thus, we can pose (5) in the following equivalent state-space form:

$$\min_{q_t^i, \Delta q_t^i} \quad \sum_{t \in \mathcal{T}_k} \bar{\varphi}_t := \sum_{t \in \mathcal{T}_k} \sum_{i \in \mathcal{S}} \int_0^{q_t^i} p_t(q, b_t^i) dq \tag{7a}$$

s.t.

$$q_{t+1}^i = q_t^i + \Delta q_t^i,\ i \in \mathcal{S},\ t \in \mathcal{T}_k^{-} \tag{7b}$$

$$\sum_{i \in \mathcal{S}} q_t^i \geq \sum_{j \in \mathcal{C}} d_t^j,\ t \in \mathcal{T}_k \tag{7c}$$

$$-\left(\overline{q}^i - \underline{q}^i\right) \leq \Delta q_t^i \leq \left(\overline{q}^i - \underline{q}^i\right),\ i \in \mathcal{S}, t \in \mathcal{T}_k^{-} \tag{7d}$$

$$\underline{q}^i \leq q_t^i \leq \overline{q}^i,\ i \in \mathcal{S}, t \in \mathcal{T}_k \tag{7e}$$

$$q_k^i = \text{given},\ i \in \mathcal{S}. \tag{7f}$$

The variables Δq_t^i are the generation ramp increments that are bounded by $\pm(\overline{q}^i - \underline{q}^i)$, the maximum generation ramp that is physically possible. Since problems (7) and (5) are equivalent, their feasible sets are the same. The multipliers of the constraints (7c) are the prices $\bar{p}_t$.

The solution of the unconstrained market clearing problem represents the *ideal* performance for the market (in the absence of ramping constraints). We now consider the dynamically *constrained* market clearing problem:

$$\min_{q_t^i,\Delta q_t^i} \sum_{t\in\mathcal{T}_k} \varphi_t := \sum_{t\in\mathcal{T}_k}\sum_{i\in\mathcal{S}} \int_0^{q_t^i} p_t(q,b_t^i)dq \quad (8a)$$

$$\text{s.t. } q_{t+1}^i = q_t^i + \Delta q_t^i,\ i\in\mathcal{S}, t\in{\mathcal{T}_k}^- \quad (8b)$$

$$\sum_{i\in\mathcal{S}} q_t^i \geq \sum_{j\in\mathcal{C}} d_t^j,\ t\in\mathcal{T}_k \quad (8c)$$

$$-\underline{r}^i \leq \Delta q_t^i \leq \overline{r}^i,\ i\in\mathcal{S}, t\in{\mathcal{T}_k}^- \quad (8d)$$

$$\underline{q}^i \leq q_t^i \leq \overline{q}^i,\ i\in\mathcal{S}, t\in\mathcal{T}_k \quad (8e)$$

$$q_k^i = \text{given},\ i\in\mathcal{S}. \quad (8f)$$

The multipliers for the constraint (8c) are the prices $p_t \geq 0$. In this formulation, the ramps are bounded by $\underline{r}^i, \overline{r}^i \leq (\overline{q}^i - \underline{q}^i)$, respectively. This constrains the dynamic response of the generators. As before, we note that the bidding parameters b_t^i enter only the cost function and thus do not affect the feasible set. In this case, however, the dynamic constraints introduce time coupling because the ramp constraints might become active. Consequently, the feasible set does depend on the current state q_k^i. Accordingly, we denote the feasible set of this problem as $\Omega^{ISO}(q_k^{\mathcal{S}}, d_{\mathcal{T}_k})$, where $q_k^{\mathcal{S}} = \{q_k^1, ..., q_k^S\}$.

The constrained social welfare is denoted as $\sum_{t\in\mathcal{T}_k}\varphi_t$ with $\varphi_t \geq 0$ since $b_t^i, q_t^i \geq 0,\ t\in\mathcal{T}_k$. It is straightforward to prove that $\sum_{t\in\mathcal{T}_k}\varphi_t \geq \sum_{t\in\mathcal{T}_k}\bar{\varphi}_t$ since $\Omega^{ISO}(q_k^{\mathcal{S}}, d_{\mathcal{T}_k}) \subseteq \Omega_{UNC}^{ISO}(d_{\mathcal{T}_k})$. In other words, the performance of the constrained clearing problem is bounded by that of the unconstrained counterpart. We also have the following property, proven by Zavala and Anitescu (2011).

Property 2.3. For fixed $b_t^i \geq 0$, the point social welfare φ_t evaluated at a solution of problem (8) and $\bar{\varphi}_t$ evaluated at a solution of (5) satisfy $\varphi_t \geq \bar{\varphi}_t > 0,\ t\in\mathcal{T}_k$.

We now formally define the *market efficiency* as

$$\eta_k := \frac{\bar{\varphi}_k}{\varphi_k},\ \forall\, k. \quad (9)$$

By definition and from Proposition 2.3, we have that $\eta_k \in [0,1]$. The case where $\eta_k = 1$ is achieved if $\varphi_k = \bar{\varphi}_k$. This case implies that the prices p_k are close to those of the unconstrained market clearing problem $\bar{p}_k$, which represents the ideal market performance. The case where $\eta_k = 0$ occurs if the constrained social welfare diverges to infinity. This case occurs when the future demands cannot be met given the current states the generators and the ramping constraints. This implies that the prices p_k diverge from $\bar{p}_k$ (i.e., a small change in demand leads to large changes in price). One can show that the efficiency and price difference between the constrained and unconstrained games can be bounded by the magnitude of the ramp limits. This can be done using the following Lipschitz property (see Zavala and Anitescu (2011)).

Property 2.4. If at a solution of the game (3) and (8), each of the optimization problems satisfy LICQ and the prices $p_t,\ t\in\mathcal{T}$ and the production $q_t^i,\ t\in\mathcal{T},\ i\in\mathcal{S}$ values are large enough, then the solution is locally stable and is a Lipschitz continuous function of the game data.

3. IMPLEMENTATION ISSUES

To represent the game given by (3) and (8) in abstract form, we define the market states x_k as the set of quantities q_k^i and prices p_k and define the aggregated vector over the set $\mathcal{T}_k$ as $x_{\mathcal{T}_k} := \{x_k, ..., x_{k+T}\}$. The controls u_k are defined as the set of ramps for all suppliers $\Delta q_k^i, i\in\mathcal{S}$ with $u_{\mathcal{T}_k} = \{u_k, ..., u_{k+T-1}\}$. The bidding increments Δb_k^i are interpreted as the supplier controls and are denoted as w_k^i, and we define $w_k := \{w_k^1, ..., w_k^S\}$. We define the disaggregated supplier vectors $w_{\mathcal{T}_k}^i, i\in\mathcal{S}$, and the total aggregated vector $w_{\mathcal{T}_k}$. The bidding states b_k^i are interpreted as the supplier states z_k with aggregated vector $z_{\mathcal{T}_k}$. We include the problem data over the horizon (e.g., the demands) in the aggregated vector $m_{\mathcal{T}_k}$. We define the abstract dynamic system as

$$(x_{k+1}, z_{k+1}) = \psi_k(x_k, z_k, u_k, w_k),\ \forall\, k\geq 0. \quad (10)$$

We can eliminate the states x_k, z_k by forward propagation of (10). With this, we can express the supplier and market clearing problem entirely in terms of the controls and initial state conditions. We thus have the supplier problem,

$$\min_{w_{\mathcal{T}_k}} \sum_{t\in\mathcal{T}_k} \phi_t^i(w_t^i, u_t) \quad (11a)$$

$$\text{s.t. } w_{\mathcal{T}_k}^i \in \Omega^i, \quad (11b)$$

for $i\in\mathcal{S}$ and the constrained market clearing problem,

$$\min_{u_{\mathcal{T}_k}} \sum_{t\in\mathcal{T}_k} \varphi_t(u_t, w_t) \quad (12a)$$

$$\text{s.t. } u_{\mathcal{T}_k} \in \Omega^{ISO}(x_k, m_{\mathcal{T}_k}). \quad (12b)$$

Since the decisions of the players do not affect each others feasible sets, the resulting game is a pure Nash equilibrium problem (see Facchinei and Kanzow (2007)).

For implementation, the game given by (11) and (12) can be solved over a receding horizon: At time k we use the forecast data $m_{\mathcal{T}_k}$ (e.g., demands $d_t^j,\ t\in\mathcal{T}_k = \{k..k+T\}$) and the current states x_k, z_k. We solve the game (11) and (12) over the horizon $\mathcal{T}_k$ to obtain $u_{\mathcal{T}_k}^*, w_{\mathcal{T}_k}^*$. From these sequences, we extract only the first actions $u_k \leftarrow u_k^*, w_k \leftarrow w_k^*$. The system will evolve from its current state x_k, z_k into the states x_{k+1}, z_{k+1} according to the model (10). In the nominal case (no forecast errors in the data $m_{\mathcal{T}_k}$), the state will evolve as predicted. At the next step $k+1$, we introduce feedback in the market by shifting the horizon of the game to obtain $\mathcal{T}_{k+1} \leftarrow \{k+1..k+T+1\}$ and use the new state x_{k+1}, z_{k+1} as initial conditions. The new data $m_{\mathcal{T}_{k+1}}$ is forecast and the game problem is solved to obtain the new decisions u_{k+1}, w_{k+1}. This approach generates the feedback law $(u_k, w_k) = h(x_k, z_k, m_{\mathcal{T}_k})$.

Note that, even in the nominal case, feedback is required because the horizon T is usually finite (i.e., at time k it is not possible to foresee demands beyond time $k + T$). This implementation strategy that solves the game over a receding horizon is intuitive but it is not used in practice. This might be because of practical constraints on information exchange and decision times.

The current strategy used in practice iterates *once* between the suppliers and the ISO in a distributed manner (see Ott (2003); Baldick et al. (2005)). Here, each supplier guesses the ISO states (e.g., prices) or, implicitly, its decisions.

This can be done, for instance, by using price forecasting. The guess is denoted by $u^{\ell}_{\mathcal{T}_k}$, where ℓ is an iteration counter. The suppliers compute bidding parameters $w^{\ell}_{\mathcal{T}_k}$ by solving (11). These are sent to the ISO to solve the market clearing problem (12) to update the decisions $u^{\ell+1}_{\mathcal{T}_k}$. This strategy can be interpreted as a single Jacobi-like iteration (see Facchinei and Kanzow (2007)).

The Jacobi iterate $u^{\ell+1}_{\mathcal{T}_k}, w^{\ell+1}_{\mathcal{T}_k}$ is feasible but not optimal for the game. Feasibility follows since the suppliers decisions $w_{\mathcal{T}_k}$ do not enter the feasible set $\Omega^{ISO}(\cdot,\cdot)$ and since the supplier problems always have a feasible solution for any feasible decision of the ISO $u_{\mathcal{T}_k}$. This suboptimal strategy is an *incomplete gaming* strategy between the suppliers and the ISO. A key observation is that *the resulting incomplete gaming error generated at each step* results in suboptimal control actions (u_k, w_k) that are propagated forward in time through the dynamic system (10). This introduces additional error dynamics into the market that can lead to instability. For instance, the suboptimal gaming solution u_k, w_k obtained at time k might place the generators at a future state x_{k+1}, w_{k+1} from which the future demands $\{d^j_{k+1}, ..., d^j_{k+1+T}\}$ can only be in a suboptimal manner (e.g., using expensive generators) or not reached at all, leading to load shedding.

4. NOMINAL STABILITY

Stability, in the context of wholesale electricity markets, should reflect strong fluctuations and divergence of prices. Traditional control-theoretic stability analysis tools are not directly applicable in this context because the market is inherently dynamic and does not exhibit a natural equilibrium for the states. While it is possible to design market clearing procedures (these can be viewed as *market controllers*) that artificially introduce equilibria (i.e., by enforcing periodicity in some form), this strategy can constrain and degrade market performance. New stability analysis tools are thus needed to enable a systematic design, analysis, and implementation of *robust* and stabilizing market clearing procedures that can sustain market manipulation and strong dynamic variations of demands and renewable supply. In this section, we take a first step toward this goal by making use of a market-specific Lyapunov stability framework.

We can express the market efficiency as an implicit function of the states of the form, $\eta_k(x_k, z_k)$ or η_k for shorthand notation. Here, we use the following definition of market stability.

Definition 4.1. The market system defined by the game (11) and (12) is said to be *stable* if, given $\eta_0 \in \Omega^{\eta}(\epsilon) := \{\eta \,|\, \eta \geq \epsilon\}$ with $\epsilon \in [0,1]$, there exist sequences u_k, w_k such that $\eta_k \in \Omega^{\eta}(\epsilon),\; k = 0..\infty$.

Here, ϵ is an *efficiency threshold value*. This can be selected, for instance, based on acceptable price deviations. We note that efficiency is a state derived from the system physical states. From (9) and (8a), we can see that the states of ISO and of the suppliers x_k, z_k can be detected through the efficiency η_k since they both appear in the cost function φ_k.

The market efficiency implicitly sets a measure of stability for the prices. We propose to measure price stability as the distance between the prices of the constrained and unconstrained market clearing problems $|p_t - \bar{p}_t|$. Having such a relative measure is important since high efficiencies do not necessarily imply large prices and viceversa.

We now define the *summarizing market state*:

$$\delta_{k+1} := (1 - (\eta_{k+1} - \epsilon)) \cdot \delta_k,\; k = 0..\infty, \qquad (13)$$

with initial conditions $\delta_0 := (1 - (\eta_0(x_0, z_0) - \epsilon)) \geq \alpha > 0$. We can also use $\delta_0 := (1 - (\eta_0(x_0, z_0) - \epsilon)) \cdot \mu$ with $\mu > 0$ as long as $\eta_0 \geq \epsilon$. If $\eta_k \geq \epsilon$, $k = 0..\infty$, then for any $\alpha > 0$ such that $\delta_0 \geq \alpha$, there exists $\kappa \geq 0$ such that $\delta_k \to \kappa$ for $k \to \infty$. In other words, the summarizing market state has a stable origin. Stability of this origin implies market stability in the sense of Definition 4.1. On the other hand, if at any step we have $\eta_k < \epsilon$, the summarizing market state will increase. Subsequent violations of the efficiency threshold will make the summarizing state diverge from the origin.

We note that the efficiency can be detected through the summarizing states δ_{k+1}, δ_k. Consequently, the states x_k, z_k are detectable. This implies that the summarizing state δ_k is controllable.

For clarity, we summarize the sequence of dependencies as follows.

- The states x_k, z_k and the data $m_{\mathcal{T}_k}$, define $\eta_k(x_k, z_k)$ and δ_k.
- The control actions can be computed to give the $(u_k, w_k) = h(x_k, z_k, m_{\mathcal{T}_k}) := \tilde{h}(\delta_k)$.
- The states evolve as (10)
$$\begin{aligned}(x_{k+1}, z_{k+1}) &= \psi(x_k, z_k, h(x_k, z_k, m_{\mathcal{T}_k})) \\ &= \tilde{\psi}(x_k, z_k, m_{\mathcal{T}_k}).\end{aligned}$$
This defines $\eta_{k+1}(\tilde{\psi}(x_k, z_k, m_{\mathcal{T}_k}))$.
- The summarizing state evolves as
$$\begin{aligned}\delta_{k+1} &= \Big(1 - \Big(\eta_{k+1}(\tilde{\psi}(x_k, z_k, m_{\mathcal{T}_k})) - \epsilon\Big)\Big) \cdot \delta_k \\ &:= f(\delta_k, \tilde{h}(\delta_k)) := \tilde{f}(\delta_k).\end{aligned}$$

Using this basic set of definitions, we now illustrate how to establish sufficient stability conditions for a given market design. In addition, we demonstrate that the current market design given by the incomplete solution of the game (11) and (12) is not stabilizing.

We propose to extend the market clearing problem (12) by making use of the definition of the summarizing state as follows.

$$\min_{u_{\mathcal{T}_k^-}} \sum_{t \in \mathcal{T}_k{}^-} (\delta_{t+1} - \delta_t) \qquad (14a)$$

$$\text{s.t. } u_{\mathcal{T}_k} \in \Omega^{ISO}(x_k, m_{\mathcal{T}_k}) \qquad (14b)$$

$$\delta_{t+1} = (1 - (\eta_{t+1} - \epsilon)) \cdot \delta_t,\; t \in \mathcal{T}_k{}^- \qquad (14c)$$

$$\eta_t \geq \epsilon,\; t \in \mathcal{T}_k \qquad (14d)$$

$$\delta_k = \text{given}. \qquad (14e)$$

The objective function of this market clearing problem will be used as a *summarizing market function*, which we define as

$$V_T(\delta_k, m_{\mathcal{T}_k^-}) := -\sum_{t \in \mathcal{T}_k} (\delta_{t+1} - \delta_t) = \delta_k - \delta_{k+T}. \qquad (15)$$

A crucial observation is that the summarizing market function can be used as a Lyapunov function that we can use to establish market stability (or, equivalently, stability of the origin of δ_k). We first make the following definition.

Definition 4.2. A function $V_T(\delta_k, m_{\mathcal{T}_k})$ is a Lyapunov function for system $\delta_{k+1} = f(\delta_k, \tilde{h}(\delta_k))$ if (1) it is positive definite: in a region Ω containing the origin if for $\delta_k \in \Omega$ we have $V_T(\delta_k, m_{\mathcal{T}_k}) \geq 0$ for $\delta_k \geq 0$, $\forall k$, and (2) it is nonincreasing along the trajectory: $\Delta V_T(\delta_k, m_{\mathcal{T}_k}) := V_T(\delta_{k+1}, m_{\mathcal{T}_{k+1}}) - V_T(\delta_k, m_{\mathcal{T}_k}) \leq 0$, $\forall k$.

4.1 Infinite Horizon

We now establish stability following the traditional approach of using the cost function of the controller (in this case market clearing problem) as a Lyapunov function (see Mayne et al. (2000)). We first consider the nominal case in which the data $m_{\mathcal{T}_k}$ can be forecast perfectly.

Theorem 4.3. If the game given by (11) and (14) with $T = \infty$ has a feasible solution $\forall\ k$, then the market is stable.

Proof: From feasibility of (14d) we have that $(\delta_t - \delta_{t+1}) \geq 0$, $t \in \mathcal{T}_k^-$ so that $V_T(\delta_k, m_{\mathcal{T}_k}) = \sum_{t \in \mathcal{T}_k^-} (\delta_t - \delta_{t+1}) \geq 0$. Consequently, positive definiteness follows. To prove that the function is nonincreasing, we compare the cost functions of two consecutive problems generating two trajectories δ_t^k, $t \in \{k..k+T\}$ and δ_t^{k+1}, $t \in \{k+1..k+1+T\}$ with $T = \infty$, $\delta_k^k = \delta_k$ and $\delta_{k+1}^{k+1} = \delta_{k+1}$. We then have

$$\begin{aligned}
\Delta V_T(\delta_k) &= V_\infty(\delta_{k+1}, m_{\mathcal{T}_{k+1}}) - V_\infty(\delta_k, m_{\mathcal{T}_k}) \\
&= \sum_{t=k+1}^{\infty} (\delta_t^{k+1} - \delta_{t+1}^{k+1}) - \sum_{t=k}^{\infty} (\delta_t^k - \delta_{t+1}^k) \\
&= \left(\delta_{k+1} - \delta_\infty^{k+1}\right) - \left(\delta_k - \delta_\infty^k\right) \\
&= -\left(\delta_k - \delta_{k+1}\right).
\end{aligned}$$

The last equality follows from Bellman's principle of optimality. This enables us to establish $\delta_\infty^{k+1} = \delta_\infty^k$. From feasibility we have that $\delta_{k+1} \leq \delta_k$, the cost function, is a Lyapunov function. Consequently, the summarizing state has a stable origin, and the market is stable. □

We have established that the decay of the summarizing function is a sufficient condition for market stability. A crucial ingredient of our analysis is the need to incorporate the stabilizing constraint (14d). The feasible set of the market clearing problem now depends on the bidding states of the suppliers. A consequence is that the ISO and the suppliers might need to iterate several times (e.g., in a Jacobi manner) to be sure of obtaining a feasible solution to the game. Consequently, the existing market implementation where a single iterate is performed between the ISO and the suppliers *cannot be guaranteed to be stable* in the sense of Definition 4.1. The reason is that not every set of bidding parameters can be guaranteed to lead to a market clearing solution satisfying the stabilizing constraint. In other words, the current market design does not enable the ISO to correct the bidding quantities to stabilize the market. Hence, the market is more prone to be manipulated and destabilized by the suppliers if these do not have appropriate means to guess the ISO decisions (e.g., by price forecasting). Finding a feasible solution to the game (11) and (14) avoids these problems.

We note that the stabilizing constraint (14d) can also be monitored outside the market clearing problem to establish properties in terms of the original game (11) and (12).

4.2 Finite Horizon

An issue arising in implementation is the fact that the market clearing problem is solved over a finite receding horizon $\mathcal{T}_k = \{k..k+T\}$. Hence, even if the infinite horizon game is stable, the solution of the receding horizon game cannot be guaranteed to be stable. To establish stability conditions for this case, we define the following bound:

$$\Xi_k^1 := |V_T(\delta_{k+1}, m_{\mathcal{T}_{k+1}}) - V_{T-1}(\delta_{k+1}, m_{\mathcal{T}_k})| \quad (16a)$$

and make the following assumption

Assumption 4.4. The horizon T is sufficiently large such that there exists a finite $\alpha_T \geq 0$ satisfying $\Xi_k^1 \leq \alpha_T$, $\forall k$.

Note that $\Xi_k^1 \to 0$ as $T \to \infty$ since the cost function $V(\cdot, \cdot)$ is positive definite.

Theorem 4.5. Assume that the game defined (11) and (14) has a feasible solution $\forall\ k$ under the horizon T. If,

$$\Xi_k^1 \leq (\delta_k - \delta_{k+1}), \forall\ k, \quad (17)$$

and Assumption 4.4 holds, then the market is stable.

Proof: From feasibility, the cost function is positive definite. To prove that it is nonincreasing under (17) we establish the following:

$$\begin{aligned}
&V_T(\delta_{k+1}, m_{\mathcal{T}_{k+1}}) - V_T(\delta_k, m_{\mathcal{T}_k}) \\
&\quad = -(\delta_k - \delta_{k+1}) + V_T(\delta_{k+1}, m_{\mathcal{T}_{k+1}}) - V_{T-1}(\delta_{k+1}, m_{\mathcal{T}_k}) \\
&\quad \leq 0.
\end{aligned}$$

From feasibility, we have that the lower bound $0 \leq (\delta_k - \delta_{k+1})$. An upper bound is given as follows. The difference between two terms in the right-hand side is bounded by the positive quantity α_T given in Assumption (4.4). In addition, since $V(\cdot, \cdot)$ is positive definite and nonincreasing the term in the left-hand side is negative. Rearranging terms in the above inequality, we have

$$(\delta_k - \delta_{k+1}) \leq V_T(\delta_k, m_{\mathcal{T}_k}) - V_T(\delta_{k+1}, m_{\mathcal{T}_{k+1}}) + \alpha_T.$$

The term $(\delta_k - \delta_{k+1})$ is bounded above by a positive quantity. This implies that $\delta_{k+1} \leq \delta_k$. Consequently, the sequence $\{\delta_k\}$ is bounded, and the conclusion follows. □

We note that the term $(\delta_k - \delta_{k+1})$ introduces some inherent robustness to the market design. This enables the market to sustain a certain level of errors introduced by finite horizons, suboptimal solutions, forecast errors, and so on. The effect of forecast errors and suboptimal solutions is analyzed in the following section.

5. ROBUST STABILITY

To analyze robustness properties, we consider the case in which the game data (e.g., demand, wind) cannot be forecast perfectly. To account for this case, the *true* value of the forcing at time k will be denoted as m_k and the trajectory over the horizon $\mathcal{T}_k$ as $m_{\mathcal{T}_k}$. The forecast error trajectory generated starting at time k will be denoted as $\varepsilon_{\mathcal{T}_k} := \{\varepsilon_k(k), ..., \varepsilon_{k+T}(k)\}$ with $\varepsilon_k(k) = 0$ since the data at the current time k is assumed to be known. The symbol (k) is used to reflect the fact that the error trajectory is generated at time k using the most recent information. The forecast trajectory can be expressed as $\bar{m}_{\mathcal{T}_k} := m_{\mathcal{T}_k} + \varepsilon_{\mathcal{T}_k}$.

The forecast errors at time k, $\varepsilon_{\mathcal{T}_k}$, generate control actions that drive the summarizing state from δ_k to $\delta_{k+1} = f(\delta_k, \tilde{h}_k(\delta_k, \bar{m}_{\mathcal{T}_k}))$. This last state differs from the error-free state $\bar{\delta}_{k+1} = f(\delta_k, \tilde{h}_k(\delta_k, m_{\mathcal{T}_k}))$. The objective is to establish conditions under which the summarizing cost function with forecast errors still represents an improvement over the current cost. We follow the approach proposed in Muske et al. (1994). We define the following error terms:

$$\Xi_k^1 := |V_{T-1}(\bar{\delta}_{k+1}, m_{\mathcal{T}_k}) - V_T(\bar{\delta}_{k+1}, m_{\mathcal{T}_{k+1}})| \tag{18a}$$

$$\Xi_k^2 := |V_T(\bar{\delta}_{k+1}, m_{\mathcal{T}_{k+1}}) - V_T(\delta_{k+1}, m_{\mathcal{T}_{k+1}})|. \tag{18b}$$

Under conditions of Property (2.4), we have that

$$\Xi_k^2 \leq L_V |\bar{\delta}_{k+1} - \delta_{k+1}| \tag{19}$$

$$\leq L_V |f(\delta_k, \tilde{h}_k(\delta_k, \bar{m}_{\mathcal{T}_k})) - f(\delta_k, \tilde{h}_k(\delta_k, m_{\mathcal{T}_k}))| \tag{20}$$

$$\leq L_V L_\delta L_h \|\varepsilon_{\mathcal{T}_k}\|. \tag{21}$$

Theorem 5.1. Assume that the game defined (11) and (14) has a feasible solution $\forall\, k$ under the horizon T and that there exists a finite $\beta \geq 0$ such that the error sequence remains bounded $\varepsilon_{\mathcal{T}_k} \leq \beta$, $\forall k$. If,

$$\Xi_k^1 + \Xi_k^2 \leq (\delta_k - \bar{\delta}_{k+1}), \forall\, k, \tag{22}$$

then the market is stable.

Proof: From feasibility, the cost function $V_T(\delta_{k+1}, m_{\mathcal{T}_{k+1}})$ is positive definite. To prove that it is nonincreasing under (22), we establish the following.

$$\begin{aligned} &V_T(\delta_{k+1}, m_{\mathcal{T}_{k+1}}) - V_T(\delta_k, m_{\mathcal{T}_k}) \\ &\quad = (\delta_k - \bar{\delta}_{k+1}) \\ &\qquad + V_{T-1}(\bar{\delta}_{k+1}, m_{\mathcal{T}_k}) - V_T(\bar{\delta}_{k+1}, m_{\mathcal{T}_{k+1}}) \\ &\qquad + V_T(\bar{\delta}_{k+1}, m_{\mathcal{T}_{k+1}}) - V_T(\delta_{k+1}, m_{\mathcal{T}_{k+1}}) \\ &\quad \leq -(\delta_k - \bar{\delta}_{k+1}) + \Xi_k^1 + \Xi_k^2. \end{aligned}$$

The last inequality follows from the stability condition (22). The function is nonincreasing. We note the appearance of the extra term $\bar{\delta}_{k+1}$. To account for this, we compute the summation of the above inequality over $j = 0, ..., J$ to obtain

$$\begin{aligned} &V_T(\delta_{k+1+J}, m_{\mathcal{T}_{k+1+J}}) - V_T(\delta_k, m_{\mathcal{T}_k}) \\ &\quad \leq -\sum_{j=0}^{J} \delta_{k+j} + \sum_{j=0}^{J} \bar{\delta}_{k+1+j} + \sum_{j=0}^{J} \Xi_{k+j}^1 + \sum_{j=0}^{J} \Xi_{k+j}^2 \\ &\quad \leq 0. \end{aligned}$$

Since $\delta_k \geq 0$, we have that $0 \leq \sum_{j=0}^{J} \delta_{k+j}$. The second term in the right-hand side is bounded and positive since, from feasibility, we have $\bar{\delta}_{k+1+j} \leq \delta_{k+j}$, $\forall j, k$. The last two terms are positive and bounded as well by α_T and β, respectively. The function $V(\cdot, \cdot)$ is positive definite and nonincreasing, so the difference in the left-hand side is negative. Consequently, the sum $\sum_{j=0}^{J} \delta_{k+j}$ remains bounded. If we extend $J \to \infty$, the conclusion follows. □

Using the same construct, we can establish stability conditions for the case in which there is incomplete gaming at each step. This will introduce an additional error $\hat{\varepsilon}_k$ to the control action $\tilde{h}(\delta_k, \bar{m}_{\mathcal{T}_k})$ that will move the state from δ_k to δ_{k+1}. In this case, we have that

$$\begin{aligned} \Xi_k^2 &\leq L_V |\bar{\delta}_{k+1} - \delta_{k+1}| \\ &\leq L_V |f(\delta_k, \tilde{h}_k(\delta_k, \bar{m}_{\mathcal{T}_k})) - f(\delta_k, \tilde{h}_k(\delta_k, m_{\mathcal{T}_k}) + \hat{\varepsilon}_k)| \\ &\leq L_V L_\delta \left(L_h \|\varepsilon_{\mathcal{T}_k}\| + \|\hat{\varepsilon}_k\|\right). \end{aligned}$$

Since this bound is larger, the robust stability threshold (22) is narrower.

We note that the stability conditions (17) and (22) cannot be guaranteed to hold for the proposed market design. In other words, they establish only inherent robustness properties. We also note that, because of forecast errors, the solution given by the game at k can be guaranteed only to satisfy the current demand but not the demands beyond $k+1$. Consequently, the cost function of the game given by $V_T(\delta_{k+1}, \bar{m}_{\mathcal{T}_{k+1}})$ cannot be used as a Lyapunov function. We therefore use the cost function of the game with no forecast errors $V_T(\delta_{k+1}, m_{\mathcal{T}_{k+1}})$ whose solution does satisfies the demands, and we deal with the forecast errors implicitly through the initial state δ_{k+1}.

For the finite horizon case with perfect forecast, we can note that the term Ξ_k^1 is still present and depends on the data $m_{\mathcal{T}_k} := \{m_k, ..., m_{k+T}\}$. Consequently, *even if the forecast is perfect*, a strong change in the data m_{k+T} to m_{k+T+1} can break the stability condition (17) if the horizon is not long enough. This situation can arise, for instance, from a steep change in wind supply. This explains why the horizon should be sufficiently large so that the bound α_T is as small as possible and the market is more robust.

Consider the limiting case in which the ramps approach the maximum production limits. We can show that the stability threshold is the same as that of the infinite horizon problem.

Theorem 5.2. Assume that the ramp limits are given by $\underline{r}^i, \bar{r}^i = (\bar{q}^i - \underline{q}^i)$. Then, the market given by the game (11) and (14) is stable with $\Xi_k^2 = \Xi_k^1 = 0$, $\forall k$.

Proof: In the absence of ramping constraints, the game problem is decoupled in time. Consequently, the optimal controls u_k, w_k are invariant to the forecast errors at the future times $k+1, ..., k+T$ and to the length of the horizon T. If we solve the game at k with $\bar{m}_{\mathcal{T}_k}$ we have that $\bar{\delta}_{k+1} = \delta_{k+1}$ since $\varepsilon_k(k) = 0$, $\forall k$. With this we have $\Xi_k^2 = 0$, $\forall k$. If we solve the game over a sequence of steps $i = 0, ..., T-1$ and collect the first terms in $V_T(\delta_{k+i}, \bar{m}_{\mathcal{T}_{k+i}})$ given by $(\delta_{k+i} - \delta_{k+1+i})$, we have that $\sum_{i=0}^{T} (\delta_{k+i} - \delta_{k+1+i}) = V_T(\delta_k, m_{\mathcal{T}_k})$, $\forall k$. We also have that we can extend the horizon and sequence as $J = T \to \infty$ so that $\Xi_k^1 = 0$, $\forall k$. Consequently, the sequence $\{\delta_k\}$ is bounded, and the market is stable. The proof is complete. □

This implies that the stability bounds of the market clearing problem are the same as those of the infinite horizon problem with perfect forecast information. In other words, when the generators are sufficiently fast, the market robustness increases, as expected. This result also implies that as the ramping capacity increases, the horizon can be made shorter.

6. CONCLUSIONS AND FUTURE WORK

In this work, we analyze robustness properties of wholesale market designs using a control-theoretic framework. We characterize robustness bounds in the presence of forecast errors and incomplete gaming. The proposed discrete time framework, however, does not enable us to understand how solution frequency (given by the time between k and $k+1$) affects robustness. Extending the analysis to continuous time can overcome this limitation. In addition, such an analysis can provide insights into high-frequency effects such as the need of operating reserves and their effect on robustness and the emergence of multi scale markets (e.g., forward and real time). Another potential extension is the incorporation of stochastic or robust market clearing formulations with stronger robustness guarantees. Another extension is the development of strategies to choose the stability threshold value ϵ in a more systematic way.

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