Paleoclassical transport in low collisionality toroidal plasmas

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Radial electron heat transport in a low collisionality plasma confined in an axisymmetric toroidal magnetic field geometry is hypothesized to be caused by the paleoclassical collisional processes of parallel electron heat conduction and radial magnetic field diffusion. The electron temperature is Maxwellianized and the electron temperature equilibrated over a long length \(L\) (> > the poloidal periodicity half-length \(\pi R_0 q\)) along helical magnetic field lines that are diffusing radially with magnetic field diffusion \(D_\eta \equiv \eta_\eta^{\nu_e}/\mu_0 \approx \nu_e (c/\omega_p)^2\). This induces a radial electron heat diffusivity \(\chi^\eta c^2\) that is a multiple \(M \approx L/(\pi R_0 q) \sim 10 >> 1\) of the magnetic field line diffusivity: \(\chi^\eta c^2 \equiv (3/2) M D_\eta\). New paleoclassical model developments in this paper include: full axisymmetric toroidal magnetic geometry, evolution of toroidal, poloidal, and helical magnetic fluxes, effects of temporally varying magnetic fluxes, introduction of the electron guiding center radial diffusion effects induced by poloidal magnetic flux diffusion into the electron drift-kinetics via a Fokker-Planck procedure, and determination of both axisymmetric and the usually dominant helically resonant paleoclassical transport. The many areas where the paleoclassical electron heat diffusivity and heat flux provide interpretations of toroidal plasma experimental results are summarized.

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I. INTRODUCTION

A new “paleoclassical transport” paradigm has recently been hypothesized [1]; it provides interpretations for most “anomalous” electron heat diffusion observed in low collisionality toroidal plasmas (for \(T_e \lesssim 2\) keV). The key new physical point is that as magnetic field lines diffuse radially (with \(D_\eta \approx \eta_\eta^{\nu_e}/\mu_0 \approx \nu_e (c/\omega_p)^2\)), they carry with them electron heat equilibrated over a long parallel length \(L\), which is the minimum of the electron collision length \(\lambda_e\) and a maximum effective field line length. Because \(L\) is much longer than the poloidal periodicity length \(\pi R_0 q\), the induced radial electron heat diffusivity is of order \(M \approx L/(\pi R_0 q) \sim 10 >> 1\) times \(D_\eta\). Since the initial paleoclassical model, which used a sheared slab magnetic field model, required a number of interpretations to represent toroidal geometry effects, this paper develops the model further by calculating the paleoclassical radial electron heat heat flux in an axisymmetric toroidal geometry — to make it applicable to most types of toroidal plasma experiments.

In a 1970 paper [2] Grad and Hogan made a prophetic statement: “plasma diffusion is a complex phenomenon with at least two distinct diffusive time scales nonlinearly coupling field diffusion (the skin effect), plasma diffusion, plasma convection, and geometrical effects . . .” The magnetic field diffusion is the faster time scale phenomenon because the magnetic field diffusivity \(D_\eta\) is larger than the classical or neoclassical diffusivity — because the electromagnetic skin depth \(\delta_e\) is usually larger than the electron gyroradius or banana width. Thus, magnetic field lines diffuse radially faster than collisions cause electrons to diffuse relative to them. The Grad and Hogan paper and subsequent papers by Pao [3, 4], as well as the standard review papers [5, 6] for neoclassical Pfirsch-Schlüter and banana-plateau transport, determined axisymmetric plasma transport in the presence of magnetic field diffusion. However, in these and subsequent papers the field diffusion effects were only considered within the context of plasma fluid models; in particular, the magnetic field diffusion did not influence the kinetic analysis.

The key hypothesis of the paleoclassical model is to add the effects of radial magnetic flux (field line) diffusion to electron drift-kinetic and hence neoclassical-type theory — by including spatial Fokker-Planck operators representing the effects of magnetic field diffusion on electron guiding center motion in the drift-kinetic equation. Such a Fokker-Planck treatment is appropriate because the poloidal magnetic flux traversed by an electron’s gyromotion diffuses radially due to the magnetic field diffusion, which causes the radial position of the electron’s guiding center to not be a lowest order constant of the motion as it usually is, but to become a “stochastic” (diffusing) variable — see discussion at the beginning of Section VI.

The initial paleoclassical model [1] used a sheared slab magnetic field geometry and assumed, for simplicity, the magnetic field was stationary. This paper develops the paleoclassical model for a general axisymmetric toroidal magnetic field geometry, and allows for temporally evolving magnetic fluxes, field lines. Also, the full Fokker-Planck procedure for including effects of radial advection and diffusion of such field lines in electron drift-kinetic theory is developed. Further, the kinetics of the Maxwellianization of the electron distribution along axisymmetric (poloidal) and nonaxisymmetric (helical) field lines that are diffusing radially is developed and ex-

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plored in a toroidal geometry. Finally, a ballooning-type representation for the distribution function is developed and used to preserve poloidal and helical symmetries in the paleoclassical “flute-like” transport responses in the vicinity of medium order rational surfaces \((q_* \equiv m/n)\) with \(n \leq n_{\text{max}} \simeq 1/(\pi \delta d q/dr)^{1/2} \sim 10\) — see (120).

This paper is organized as follows. Section II discusses key properties of the axisymmetric magnetic field model used throughout the paper. The following section (III) uses components of Faraday’s law to develop evolution equations for the toroidal and poloidal magnetic fluxes in terms of appropriate components of the electric field. Section IV develops a relevant (flux-surface-averaged, neoclassical plus inertia) parallel Ohm’s law. The motion and diffusion of magnetic flux surfaces and field lines, and in particular helical fluxes in the vicinity of rational surfaces, are explored in Section V. The effects of magnetic field diffusion on electron guiding center motion and the resultant drift-kinetic equation for paleoclassical processes, which includes the Fokker-Planck spatial operators representing the effects induced by the radial diffusion of poloidal magnetic flux, is developed and solved in Section VI. Thereafter, in Section VII an electron energy balance equation for paleoclassical processes is obtained and its consequences explored. Section VIII summarizes the main contributions of this paper.

II. AXISYMMETRIC MAGNETIC FIELD GEOMETRY

The initial paper on paleoclassical radial electron heat diffusion [1] used a sheared slab magnetic field model. Here, in order to more accurately describe relevant magnetic field geometries and to clarify some ambiguities that arise in the sheared slab model, a full axisymmetric magnetic field model is used. In what follows the paleoclassical model is developed for arbitrary aspect ratio \((A \equiv R_0/r \equiv 1/e)\) where \(R_0, r \) are the major, minor radii of the torus to facilitate application of the paleoclassical model to most types of axisymmetric toroidal plasmas — large aspect ratio tokamaks \((A >> 1)\), and regions of spherical tokamaks \((\text{STs}, A \gtrsim 1)\), spheromaks, and reversed field pinches \((\text{RFPs})\) where the squares of the local inverse aspect ratio and ratio of the poloidal magnetic field \((B_p)\) to the toroidal magnetic field \((B_t)\) are small: \(c^2 B_p^2/B_t^2 << 1\). However, the theory is developed from a tokamak perspective in that the usual arbitrary aspect ratio tokamak variables are used throughout. Approximate forms for large aspect ratio, nearly circular cross-section tokamaks are indicated at the end of many equations after an approximate equality \((\approx)\).

The paleoclassical transport model is concerned with diffusion of magnetic flux. The first question that arises is: diffusion relative to what coordinates? Hirschman and Jardin [7] have shown that for axisymmetric toroidal plasmas with \(c^2 B_p^2/B_t^2 << 1\) the toroidal magnetic flux \(\psi_t(x, t)\) is less mobile than the poloidal magnetic flux \(\psi(x, t)\) — because \(\psi\) is caused by the self-consistent parallel (~ toroidal) currents flowing in the plasma while \(\psi_t\) is due predominantly to currents flowing poloidally in external coils. (For an approach that allows both poloidal and toroidal fluxes to be evolving, and possibly related by three-dimensional effects, see Strand and Houlberg [8].) For such plasmas transport fluxes are calculated [5-7] relative to the (usually small) “radial grid velocity” of toroidal magnetic flux surfaces induced by the poloidal electric field. Thus, diffusion of the poloidal magnetic flux surfaces (and field lines) will be determined relative to the toroidal magnetic flux surfaces:

\[
\psi_t(p, t) \equiv \frac{1}{2 \pi} \int dS(\zeta) \cdot B_t \approx \frac{r^2 B_0}{2}, \quad \text{toroidal flux, (1)}
\]

in which \(B_0\) is the total magnetic field strength at the magnetic axis \((\psi_t = 0)\) and the surface integral is over the cross-sectional area of a given magnetic flux surface in a constant toroidal angle plane. A convenient, dimensionless, cylindrical-type radial variable is \(\rho\):

\[
\rho \equiv \sqrt{\psi_t/\psi_t(a)} \simeq r/a, \quad \text{radial coordinate, (2)}
\]

in which \(\psi_t(a)\) is the toroidal magnetic flux enclosed by the last closed flux surface in the plasma (minor radius \(a\) for a circular cross-section plasma).

The appropriate magnetic field model [6, 7] for a low \(\beta\) axisymmetric toroidal plasma can be described in terms of its toroidal \((t)\) and poloidal \((p)\) components by

\[
B = B_t + B_p = B_t e_\zeta + B_p e_\theta
\]

\[
= I \nabla \zeta + \nabla \zeta \times \nabla \psi = \nabla \psi \times \nabla (q \theta - \zeta)
\]

\[
= \nabla \times A, \quad \text{for } A \equiv \psi_t \nabla \theta - \psi \nabla \zeta. \quad (3)
\]

Here, presuming an axisymmetric ideal magnetohydrodynamic (MHD) equilibrium (i.e., one that has an isotropic pressure to lowest order and whose magnetic flux satisfies the Grad-Shafranov equation — see for example [9, 10]) \(I = I(\rho, t) \equiv RB_t \simeq B_0 R_0\) is a current function that represents predominantly the currents flowing poloidally in the external coils in which \(R \equiv 1/|\nabla \zeta| \approx R_0 + \rho a \cos \theta + O(\epsilon^2 R_0)\) is the major radius to a given point on a flux surface. Also, \(\zeta\) is the toroidal (axisymmetry, long way around the torus) angle and \(\psi\) is the poloidal magnetic flux function:

\[
\psi(\rho, t) \equiv \frac{1}{2 \pi} \int dS(\theta) \cdot B_p, \quad \text{poloidal flux, (4)}
\]

for which \(\partial \psi/\partial \rho \simeq a R_0 B_p\). Further, \(\theta\) is the poloidal (short way around torus) angle in which the magnetic field lines are straight (in the \(\psi = \text{constant plane}\) and \(q\) is the winding number or pitch (“safety factor” for kink stability) of magnetic field lines on a flux surface:

\[
q(\rho, t) \equiv \frac{\partial \psi_t}{\partial \psi} = \frac{\# \text{ toroidal transits}}{\# \text{ poloidal transits}} \simeq \frac{r B_t}{R_0 B_p}. \quad (5)
\]

For an axisymmetric magnetic field \(q(\rho, t) = q(\psi, t)\) and

\[
B \cdot \nabla \theta = I/q R^2 \simeq B_t/R_0 q = B_p/r. \quad (6)
\]
Some useful properties of this magnetic field description and coordinate system are as follows. First, note that it is not an orthogonal coordinate system since \( \nabla \psi \cdot \nabla \theta \neq 0 \). The Jacobian of the transformation from the original fixed Eulerian coordinates to the curvilinear set \( u^i = (\rho, \theta, \zeta) \) is

\[
\sqrt{g} \equiv \frac{1}{\nabla \rho \cdot \nabla \theta \times \nabla \zeta} = \frac{\partial \psi / \partial \rho}{B \cdot \nabla \theta} = \frac{\partial \psi / \partial \rho}{1/\sqrt{q} R^2} \approx raR_0. \tag{7}
\]

The volume of a magnetic flux surface whose label is \( \rho \) is

\[
V(\rho, t) = \int_0^\rho d\rho = \int_0^\rho d\rho \int_0^\pi d\theta \int_{-\pi}^\pi d\zeta \sqrt{g} = 2\pi \int_0^\rho d\rho \int_{-\pi}^\pi \sqrt{g} d\theta \approx \pi r^2(2\pi R_0). \tag{8}
\]

Hence, the partial derivative of the volume with respect to the flux surface label \( \rho \) is

\[
V' = \frac{\partial V}{\partial \rho} = 2\pi \int_{-\pi}^\pi \sqrt{g} d\theta \approx 2\pi a(2\pi r)(2\pi R_0). \tag{9}
\]

The average of an axisymmetric \( \partial f/\partial \zeta = 0 \) scalar function \( f(x, t) \) over a flux surface can be defined using a limiting process on the differential volume between a toroidal magnetic flux surface at \( \rho + \delta \rho \) and one at \( \rho \):

\[
\langle f(x, t) \rangle = \lim_{\delta \rho \to 0} \frac{\int_\rho^{\rho+\delta \rho} \int_0^\pi d\rho \int_{-\pi}^\pi d\theta \int_{-\pi}^\pi \sqrt{g} f(x, t)}{\int_\rho^{\rho+\delta \rho} \int_0^\pi d\rho \int_{-\pi}^\pi d\theta \int_{-\pi}^\pi \sqrt{g}} = 2\pi \int_{-\pi}^\pi \sqrt{g} \frac{df}{d\theta} f(x, t) \frac{\sqrt{g}}{2\pi} \approx \int_{-\pi}^\pi \frac{d\theta}{2\pi} \left[ 1 + 2 \frac{\partial \psi}{\partial \rho} \cos \theta + O(\zeta^2) \right] f(x, t). \tag{10}
\]

The flux-surface-average is an annihilator for the parallel gradient operator \( B \cdot \nabla \):

\[
\langle B \cdot \nabla f \rangle = 0, \quad \tag{11}
\]

for any function \( f(x, t) \) that is a periodic function of both \( \theta \) and \( \zeta \). For a similarly periodic vector field \( A(x, t) \), the flux-surface-average of its divergence, which is defined by

\[
\langle \nabla \cdot A \rangle \equiv \sum_i (1/\sqrt{g})(\partial/\partial u_i)(\sqrt{g} A \cdot \nabla u_i),
\]

becomes

\[
\langle \nabla \cdot A \rangle = \frac{1}{V'} \frac{\partial}{\partial \rho} \langle V' \cdot (A \cdot \nabla \rho) \rangle \equiv \frac{\partial}{\partial V} \langle A \cdot \nabla V \rangle. \tag{12}
\]

For the toroidal magnetic flux the relevant differential surface area is \( dS(\zeta) = \sqrt{g} \, d\rho \, d\theta \, \sqrt{g} \zeta \); hence using (9) and the fact that \( I = I(\rho, t) \), the toroidal magnetic flux density (flux per unit volume) becomes

\[
\frac{\partial \psi_t}{\partial V} = \frac{1}{V'} \frac{\partial \psi_t}{\partial \rho} = \frac{1}{V'} \int_{-\pi}^\pi \sqrt{g} \, d\theta \frac{B \cdot \nabla \zeta}{2\pi} = I(\rho, t) \langle R^{-2} \rangle / 2\pi \approx B_0/(2\pi R_0). \tag{13}
\]

Magnetic flux surfaces are rational or irrational depending on whether or not \( q \) is the ratio of integers \((m, n)\):

\[
q(\rho, t) = \begin{cases} m/n, & \text{rational surface,} \\ \pm m/n, & \text{irrational surface.} \end{cases} \tag{14}
\]

The irrational surfaces form a dense set while the rational surfaces are a set of measure zero and radially isolated from each other. The rational surfaces are of interest here because their (helical) magnetic field lines close on themselves after \( m \) toroidal (or \( n \) poloidal) transits.

The differential length \( dl \) along magnetic field lines can be obtained from the poloidal \( (\nabla \theta) \) projection of the equation \( dx/dl = B/B \) that defines magnetic field lines:

\[
dl = \frac{B}{B \cdot \nabla \theta} d\theta \approx R_0 q d\theta. \tag{15}
\]

A property of magnetic field lines of interest in paleo-transportal is their length. The half length of any function \( f(x, t) \) over a flux surface can be defined using a limiting process on the differential volume between a toroidal magnetic flux surface at \( \rho + \delta \rho \) and one at \( \rho \):

\[
\langle f(x, t) \rangle = \lim_{\delta \rho \to 0} \frac{\int_\rho^{\rho+\delta \rho} \int_0^\pi d\rho \int_{-\pi}^\pi d\theta \int_{-\pi}^\pi \sqrt{g} f(x, t)}{\int_\rho^{\rho+\delta \rho} \int_0^\pi d\rho \int_{-\pi}^\pi d\theta \int_{-\pi}^\pi \sqrt{g}} = 2\pi \int_\rho^{\rho+\delta \rho} \int_0^\pi d\rho \int_{-\pi}^\pi \sqrt{g} \frac{df}{d\theta} f(x, t) \frac{\sqrt{g}}{2\pi} \approx \int_{-\pi}^\pi \frac{d\theta}{2\pi} \left[ 1 + 2 \frac{\partial \psi}{\partial \rho} \cos \theta + O(\zeta^2) \right] f(x, t). \tag{10}
\]

Note that while helical field lines on rational surfaces with \( n \approx 10 > 1 \) are quite long \((> \pi R_0)\), those with low \( n \) \((\approx n^x = 1, 2)\) are short \((\approx \pi R_0)\).

Another set of properties of magnetic field lines of interest in the paleo-classical model are the radial distances between medium order \((e.g., n \approx 10 >> 1)\) rational surfaces. To determine these, one first expands \( q(\rho, t) \) in a Taylor series expansion about its value on a rational surface located at \( \rho = \rho_0 \):

\[
q(\rho, t) \approx q_0 + x q' + O(x^2), \quad \text{local } q(\rho \text{ expansion, } \tag{18}
\]

in which the following key quantities have been defined:

\[
q_0 \equiv q(\rho_0, t) \equiv m/n, \quad q \text{ value on rational surface, } \tag{19}
\]

\[
x \equiv \rho - \rho_0, \quad \text{distance from rational surface, } \tag{20}
\]

\[
q' \equiv \partial q/\partial \rho|_{\rho_0}, \quad \text{local magnetic shear variable. } \tag{21}
\]

Note that all these quantities are dimensionless — because \( \rho \) is a dimensionless radial coordinate. The distance to adjacent rational surfaces with \( m \pm 1 \) but the same \( n \) is obtained from \( 1/n = q - q_0 \approx x q' \):

\[
\Delta \approx 1/nq', \quad \text{spacing of same } n \text{ rational surfaces. } \tag{22}
\]

Here and henceforth, if \( q' \) is negative, it is replaced by \( |q'| \).

Next, consider the distance between a \( q_0 \equiv m/n \) rational surface and the nearest rational surface with \( n \leq n_{\text{max}} \). The maximum number \( n_{\text{max}} \) (typically \( \approx 10 \)) will be
derived in (120) below. Defining $q(q_{\text{max}}) = m_{\text{max}}/n_{\text{max}}$ and expanding $q(\rho) = (m_{\text{max}} n + 1)/n_{\text{max}} n$ in a Taylor series about $\rho = q_{\text{max}}$, the separation is found to be

$$\delta x \equiv \rho - q_{\text{max}} \simeq \frac{1}{n_{\text{max}} n q'}, \quad \text{minimum spacing.} \quad (23)$$

At a minimum in $q$ where $q'$ vanishes, one obtains

$$\delta x_{\text{min}} \equiv \rho - q_{\text{max}} \simeq \left( \frac{2}{n_{\text{max}} n q''} \right)^{1/2}, \quad \text{if } q'' \equiv \frac{\partial^2 q}{\partial \rho^2} |_{\rho = q_{\text{max}}}. \quad (24)$$

For $n_{\text{max}} \gtrsim 10, q' \sim 1$, and $q'' \sim 1$, all these are small fractions of the minor radius: $\Delta \sim 1/q_{\text{max}} < 1$ for $n \sim n_{\text{max}}$, and $\delta x \lesssim 1/n_{\text{max}} < 1, \delta x_{\text{min}} \lesssim 1/n_{\text{max}} < 1$.

### III. MAGNETIC FLUX EVOLUTION EQUATIONS

Equations for the evolution of the toroidal and poloidal magnetic fluxes are obtained from Faraday’s law:

$$\partial \mathbf{B}/\partial t = -\nabla \times \mathbf{E}. \quad (25)$$

Taking the $dS(\zeta)/dV = \sqrt{\gamma} d\theta \nabla \zeta \times V$ projection of this equation divided by $2\pi$, using the definition of $\partial \psi_\zeta/\partial V$ from (13), integrating over $\theta$ from $-\pi$ to $\pi$, and using common vector identities and (12), one obtains [7]

$$\frac{\partial}{\partial V} \frac{\partial \psi_\zeta}{\partial t} \bigg|_x = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{\gamma} \nabla \zeta \cdot \nabla \mathbf{E} = \left( \nabla \cdot (\nabla \zeta \times \mathbf{E}) \right) \frac{2}{\pi}.$$

Integrating this last form over volume from the magnetic axis where $V = \psi_\zeta = 0$, which remains fixed in space since $\nabla V = 0$ and hence $\partial \psi_\zeta/\partial t = 0$ there, to $V$ yields

$$\frac{\partial \psi_\zeta}{\partial t} \bigg|_x = -\frac{\mathbf{E} \cdot \nabla \zeta \times \nabla \mathbf{E}}{2\pi} = -\frac{\mathbf{E} \cdot \mathbf{B}_p q}{(\mathbf{B} \cdot \nabla \zeta)}. \quad (27)$$

Here, (13) and the fact from (5) that $\partial \psi_\zeta/\partial V = q \partial \psi/\partial V$ have been used. Thus, since $\mathbf{B}_p \equiv \nabla \zeta \times \nabla \psi$ is proportional to the covariant base vector in the poloidal direction, changes in the toroidal magnetic flux are induced by the (inductive) poloidal electric field in the plasma.

In a tokamak $(\mathbf{E} \cdot \mathbf{B}_p)$ is mostly caused by transient changes in the radial position (Shafranov shift) and shape (ellipticity, triangularity, etc.) of magnetic flux surfaces induced by changing poloidal currents in the external poloidal field shaping coils that produce changes in the toroidal magnetic flux. [These externally-induced effects can be introduced explicitly by adding a spatially constant $\partial \psi_\zeta/\partial t = V_{\text{loop}}^\zeta(t)/2\pi$ term to the right of (27).]

The induced motion of the toroidal flux surfaces is usually called the “grid velocity.” In RFPs and spheromaks inductive poloidal electric fields are often induced transiently during “helicity injection.” (There is also a small poloidal electric field induced by the parallel (to $\mathbf{B}$) plasma resistivity which causes diffusion of the toroidal magnetic flux that is of order $e^2 B_p^2/B^2 << 1$ smaller than diffusion of the poloidal magnetic flux $\psi$ and hence will be neglected — see discussion after (36) below.)

The radial grid velocity $\mathbf{u}_g$ of toroidal flux surfaces (relative to laboratory coordinates) can be determined from the equation obtained by setting the total differential of $\psi_\zeta(x, t)$ to zero $[\partial \psi_\zeta/\partial t = 0 = \partial \psi/\partial t + (\partial \psi/\partial \psi_\zeta) \partial \psi_\zeta/\partial t]$:

$$\mathbf{u}_g \cdot \nabla \psi_\zeta = \frac{d\mathbf{x}}{dt} \cdot \nabla \psi_\zeta = -\frac{\partial \psi_\zeta}{\partial t} \bigg|_x. \quad (28)$$

Flux-surface-averaging this equation and combining the last two equations yields

$$\frac{d\psi_\zeta}{dt} = \frac{\partial \psi_\zeta}{\partial t} \bigg|_x + (\mathbf{u}_g \cdot \nabla \psi_\zeta) = 0, \quad (29)$$

$$(\mathbf{u}_g \cdot \nabla \psi_\zeta) = q (\mathbf{E} \cdot \mathbf{B}_p)/(\mathbf{B} \cdot \nabla \zeta), \quad \text{grid velocity.} \quad (30)$$

Hence, the toroidal flux $\psi_\zeta$ is advected radially by the grid velocity $\mathbf{u}_g$, but conserved in a Lagrangian frame moving with $\mathbf{u}_g$. Using the fact from (5) that $\nabla \psi_\zeta = q \nabla \psi$ and the magnetic field vector identity $\mathbf{B}_p \equiv \nabla \zeta \times \nabla \psi = \mathbf{B} - i \nabla \zeta$, (30) can be rearranged to yield a relation that will be useful for the poloidal flux evolution equation:

$$(\mathbf{E} \cdot \nabla \zeta)/(R^{-2}) = (\mathbf{E} \cdot \mathbf{B})/(\mathbf{B} \cdot \nabla \zeta) - (\mathbf{u}_g \cdot \nabla \psi). \quad (31)$$

To develop an evolution equation for the poloidal flux function $\psi$, the electric field is first split into its components in and perpendicular (or cross) to the $\nabla \zeta$ direction:

$$\mathbf{E} = \left( \frac{\mathbf{E} \cdot \nabla \zeta}{|\nabla \zeta|^2} \right) \nabla \zeta + \nabla \zeta \times \mathbf{E}_\Lambda, \quad \mathbf{E}_\Lambda = \frac{\mathbf{E} \times \nabla \zeta}{|\nabla \zeta|^2}. \quad (32)$$

The $\mathbf{E}_\Lambda$ component contributes to $\partial \psi_\zeta/\partial t$ [see (26)], which has already been determined. The components of Faraday’s law (25) in the $\zeta = \text{constant}$ plane yield

$$\nabla \zeta \times \nabla \left( \frac{\partial \psi}{\partial t} - \frac{\mathbf{E} \cdot \nabla \zeta}{|\nabla \zeta|^2} \right) = 0. \quad (33)$$

Since $|\nabla \zeta|^2 = R^{-2}$, the solution of this equation is

$$\frac{\partial \psi}{\partial t} = R^2 (\mathbf{E} \cdot \nabla \zeta) - \frac{\partial \psi}{\partial t}, \quad (34)$$

in which $\partial \psi/\partial t \equiv V_{\text{loop}}^\psi(t)/2\pi$ is a (positive) constant of the spatial integration. It represents the toroidal loop voltage induced by the rate of change of the magnetic flux in the central solenoid of a tokamak — the ohmic heating transformer. (The sign is negative because the Poynting flux usually points into the plasma — to drive the toroidal current.) Multiplying by $1/R^2$, taking the flux surface average, and then dividing by $\langle R^{-2} \rangle$ yields

$$\frac{\partial \psi}{\partial t} = \frac{\langle \mathbf{E} \cdot \nabla \zeta \rangle}{\langle R^{-2} \rangle} - \frac{\partial \psi}{\partial t}. \quad (35)$$
Substituting in \((\mathbf{E} \cdot \nabla \zeta)\) from (31) finally yields the desired evolution equation for the poloidal magnetic flux:

\[
\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \langle u_y \cdot \nabla \psi \rangle = \frac{\langle \mathbf{E} \cdot \mathbf{B} \rangle}{\langle \mathbf{B} \cdot \nabla \zeta \rangle} - \frac{\partial \mathbf{B}}{\partial t}. \tag{36}
\]

The poloidal magnetic flux \(\psi\) and hence magnetic field lines move relative to the toroidal flux \(\psi_t\) [compare (29) and (36)] because of departures from ideal MHD (i.e., a nonzero parallel electric field \((\mathbf{E} \cdot \mathbf{B})\)) or a temporally changing magnetic flux in the central solenoid (i.e., \(\partial \mathbf{B}/\partial t \neq 0\)). [The parallel electric field \(\mathbf{E}_p = (\mathbf{E} \cdot \mathbf{B}) \mathbf{B} / (\mathbf{B}^2)\) also induces (see (9.24) in [6]) a small contribution to the grid velocity:

\[
\langle u_y \cdot \nabla \psi \rangle (\mathbf{E} \cdot \nabla \zeta) = \langle \mathbf{E} \cdot \mathbf{B} \rangle (1 - \rho^2 (\zeta - 1)) \sim O(\varepsilon^2 B_p^2 / B^2),
\]

which will be neglected relative to the dominant \(\langle \mathbf{E} \cdot \mathbf{B} \rangle / \langle \mathbf{B} \cdot \nabla \zeta \rangle\) term on the right of (36).] Note that only the inductive parallel electric field \(\mathbf{E}_p = -\mathbf{A} / \partial t\) contributes to \(d\psi/dt\) — because from (11) the flux surface average of a potential contribution would vanish: \(-\mathbf{A} \cdot \nabla \phi = 0\). To proceed further, plasma physics effects need to be introduced through a parallel Ohm’s law that relates the parallel electric field \(\mathbf{E} \cdot \mathbf{B} / B\) to the parallel current flowing in the plasma.

IV. PARALLEL OHM’S LAW

A parallel Ohm’s law can be obtained from the total electron momentum equation:

\[
m_e n_e \frac{dV_e}{dt} = -n_e e (\mathbf{E} + \mathbf{V}_e \times \mathbf{B}) - \nabla p_e - \mathbf{v} \cdot \pi_e + R_e. \tag{37}
\]

Here, \(d_e / dt \equiv \partial / \partial t + \mathbf{V}_e \cdot \nabla\). \(\mathbf{V}_e\) is the flow velocity of the electron fluid, \(p_e \equiv n_e T_e\) is the electron pressure, \(\mathbf{v} \cdot \pi_e \approx \nabla \cdot \pi_{ae}\) is the electron viscous force density primarily due to neo-classical parallel collisional collisions in the electron fluid, \(R_e \approx n_e e (\eta_0 J_\parallel + \eta_1 J_\perp)\) is the Coulomb collision frictional force density on the electron fluid, and the other notation is standard. Taking the parallel \((\mathbf{B} \cdot \cdot)\) projection of this equation, flux-surface-averaging assuming \(n_e\) is constant along \(\mathbf{B}\), and using (10) and (11), the resultant equation can be rearranged to yield

\[
\langle \mathbf{E} \cdot \mathbf{B} \rangle = \eta_0 \langle \mathbf{J} \cdot \mathbf{B} \rangle - \frac{1}{n_e} \langle \mathbf{B} \cdot \nabla \cdot \pi_{ae} \rangle - \frac{m_e}{e} \langle \mathbf{B} \cdot \frac{dV_e}{dt} \rangle. \tag{38}
\]

The first term indicates the effect of the parallel electrical resistivity while the other terms represent effects due to parallel electron viscosity [6] (trapped particle effects on resistivity and bootstrap current) and electron inertia.

The inertia term can be simplified assuming that: 1) the calculation is done in the rest frame of the ion fluid so that \(\mathbf{V}_e = -\mathbf{J} / n_e e\); 2) \(\mathbf{J} \cdot \mathbf{d} \mathbf{J} / dt \approx (d_e / dt) \langle \mathbf{B} \cdot \mathbf{J} \rangle / n_e e\) [i.e., \(\mathbf{J} \cdot \mathbf{d} / \partial t \cdot \mathbf{d} \mathbf{B} / \partial t\) is higher order (in \(\beta\) and hence negligible); and 3) the electron flow velocity \(\mathbf{V}_e \approx \mathbf{u}_y\) (i.e., the classical and neo-classical electron transport flows are neglected, which is justified because the classical and neo-classical electron diffusivities [5] are factors of \(\beta_e\) and \(\beta_e^2 / e^{3/2}\) smaller than the magnetic field diffusivity \(D_n\)). Then, the inertia term becomes \((m_e / n_e e^2) (d_e / dt) \langle \mathbf{J} \cdot \mathbf{B} \rangle\) and using the definitions

\[
\omega_p \equiv \sqrt{n_e e^2 / m_e e_0}, \quad \text{electron plasma frequency}, \tag{39}
\]

\[
\delta_e \equiv c / \omega_p, \quad \text{electromagnetic (em) skin depth}, \tag{40}
\]

the parallel electric field equation (38) can be written as

\[
\langle \mathbf{E} \cdot \mathbf{B} \rangle = \left( \eta_0 / \mu_0 + \delta_e^2 \frac{d_e}{dt} \right) \langle \mu_0 \mathbf{J} \cdot \mathbf{B} \rangle - \frac{1}{n_e e} \langle \mathbf{B} \cdot \nabla \cdot \pi_{ae} \rangle. \tag{41}
\]

A closure relation for the flux-surface-averaged parallel viscous force density \(\langle \mathbf{B} \cdot \nabla \cdot \pi_{ae} \rangle\) is calculated in neoclassical theory [6, 11]. It is valid on time scales after the poloidal flows have come into equilibrium (i.e., \(t > 1 / \nu_e\) — see Ref. [12]) and length scales longer than the skin depth \((\delta_e^2 \rho_e < 1)\) — so short-time-scale electron kinetics and inertial effects can be neglected. Neglecting poloidal electron heat flow and higher order poloidal flow moment effects, the parallel poloidal viscous force is [6, 11]

\[
\langle \mathbf{B} \cdot \nabla \cdot \pi_{ae} \rangle \approx \langle \mu_0 n_e e \nu_e U_{\theta e} \rangle (B^2). \tag{42}
\]

Here, \(\mu_e\) is the (collisional) electron viscous drag frequency [see (47) below] and

\[
U_{\theta e}(\psi) = \frac{V_e \cdot \nabla \theta}{B} = \frac{V_e \cdot \mathbf{B}}{B^2} - I \frac{d \Phi}{d \psi} - \frac{1}{n_e e} \frac{d p_e}{d \psi}, \tag{43}
\]

indicates the poloidal electron flow speed [6] in the presence of the lowest order parallel, \(\mathbf{E} \times \mathbf{B}\) and diamagnetic electron flows. In the ion rest frame \(V_{\theta i} = 0\) and \(U_{\theta i} = 0\); then, the radial ion force balance equation yields \(d \Phi / d \psi = -(1 / n_i q_i) \langle d p_i / d \psi \rangle\). Thus, in the ion rest frame the poloidal electron current becomes

\[
- n_e e U_{\theta e} = \langle (\mathbf{J} \cdot \mathbf{B}) + I dP / d \psi \rangle (B^2), \tag{44}
\]

in which \(P \equiv p_e + p_i\) is the total plasma pressure. Substituting this last form into (42) yields

\[
\frac{1}{n_e e} \langle \mu_0 \mathbf{J} \cdot \mathbf{B} \rangle = \eta_0 \mu_0 \nu_e \frac{d p_e}{d \psi} = \left( \eta_0 \mu_0 \nu_e \right) \frac{d \Phi}{d \psi}, \tag{45}
\]

in which the reference (perpendicular) resistivity \(\eta_0\) is written in the form of a magnetic field diffusivity,

\[
\eta_0 \approx \frac{m_e \nu_e}{n_e e^2 \mu_0} = \nu_e \delta_e^2 \approx \frac{1.4 \times 10^6 Z^2}{T_e (eV)^{3/2}} \left( \frac{\ln A}{17} \right) \frac{m^2}{s}, \tag{46}
\]

and the electron viscous drag frequency adapted from Refs. [5, 6, 11] is approximately

\[
\nu_e \approx \frac{Z^2 + \ln (1 + \sqrt{2}) f_i}{Z (1 + \nu_e^2 + \nu_e^3) f_c} \Rightarrow 1.5 f_i / f_c. \tag{47}
\]

In these formulas, \(Z \to Z_{ae} \equiv \sum_i n_i Z_i^2 / n_e\) for multiple ion species is the (effective) ion charge, \(f_c\) is the flow-weighted fraction of circulating particles [6] for which a Padé approximation is [13]

\[
f_c \approx \left( 1 - e^2 \right)^{-1/2} (1 - e^2) \sim 1 - 1.46 e^{1/2} + O(e). \tag{48}
\]
Also, the fraction of trapped particles is defined by $f_1 \equiv 1 - f_c$ and the electron collisionality parameter is

$$\nu_{ec} \equiv \frac{\nu_e}{e^{3/2}(v_{Te}/R_0q)^2} = \frac{R_0q}{e^{3/2}\lambda_e},$$

in which the electron collision length $\lambda_e$ is given by

$$\lambda_e \equiv \frac{v_{Te}}{\nu_e} \approx 1.2 \times 10^{16} \frac{[T_e(eV)]^2}{n_e Z} \left(\frac{17}{\ln \Lambda}\right) \text{ m}.$$  

(49)

Since the parallel viscous force $(\mathbf{B} \cdot \nabla \cdot \mathbf{J})|_{\parallel}$ has a term proportional to $(\mu_0 \mathbf{J} \cdot \mathbf{B})$, when it is introduced in (41) this term just adds to the parallel resistivity term. Thus, upon substituting (45) into (41) one obtains

$$\left(\frac{\mathbf{E} \cdot \mathbf{B}}{\mathbf{B} \cdot \nabla \zeta}\right) = \left(\frac{\eta_{nc}}{\mu_0} + \frac{\mu_e}{\nu_e} \right) \langle \mu_0 \mathbf{J} \cdot \mathbf{B} \rangle + \frac{\mu_e}{\nu_e} \eta_0 \frac{dP}{d\psi},$$

(51)

in which

$$\eta_{nc} \equiv \eta_0 \left(\frac{\eta_{Sp}}{\eta_0} + \frac{\mu_e}{\nu_e}\right),$$

neoclassical resistivity, 

(52)

and the parallel (Spitzer [6, 14]) electrical resistivity is approximately (including electron flow, heat flow effects)

$$\eta_{Sp} \approx \frac{\sqrt{2} + Z}{\sqrt{2} + 13Z/4},$$

Spitzer resistivity. 

(53)

This last formula is accurate (for all Z) to about 5%, which is within the intrinsic accuracy ($\sim 1/\ln \Lambda \sim 1/17 \sim 6\%$) of the Fokker-Planck collision operator used to derive it. The $\eta_{nc}$ in (52) ranges from being equal to (for $\mu_e/\nu_e << 1$) to twice as large as (for $\mu_e/\nu_e >> 1$) the most precise neoclassical resistivity results [5, 15].

Next, a form is obtained for $(\mu_0 \mathbf{J} \cdot \mathbf{B})$. Using Ampère’s law, the current density $\mathbf{J}$ embodied in the magnetic field defined in (3) is [9]

$$\mu_0 \mathbf{J} \equiv \nabla \times \mathbf{B} = (\partial I/\partial \psi) \nabla \psi \times \nabla \zeta + \nabla \psi \Delta^* \psi,$$

(54)

in which $\Delta^* \psi$ is the usual magnetic differential operator:

$$\Delta^* \psi \equiv (1/|\nabla \zeta|^2) \nabla \cdot |\nabla \zeta|^2 \nabla \psi = R^2 \nabla \cdot R^{-2} \nabla \psi.$$

(55)

Dotting this $\mu_0 \mathbf{J}$ with $\mathbf{B}$ from (3), flux surface averaging, and using $|\nabla \zeta|^2 = R^{-2}$ and (11), yields [see Eq. (7.20) and its antecedents in [5]]

$$\langle \mu_0 \mathbf{J} \cdot \mathbf{B} \rangle = I \langle |\nabla \zeta| \Delta^* \psi \rangle - (\partial I/\partial \psi) \langle |\nabla \zeta|\nabla \psi |^2 \rangle$$

$$= I \frac{\partial}{\partial \psi} \left( \frac{\nabla \mathbf{V} \cdot \nabla \psi}{R^2} \right) - \frac{\partial I}{\partial \psi} \left( \frac{|\nabla \psi|^2}{R^2} \right)$$

$$\equiv \langle \mathbf{B} \cdot \nabla \zeta \rangle \Delta^+ \psi = I \langle R^{-2} \rangle \Delta^+ \psi,$$

(56)

in which for notational simplicity

$$\Delta^+ \psi \equiv \frac{I}{(R^{-2})V} \frac{\partial}{\partial \rho} \left( \frac{|V'|}{R^2} \right) \frac{V^\prime}{I} \frac{\partial \psi}{\partial \rho} \simeq \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r \partial \psi}{\partial \rho} \right).$$

(57)

Note that $\Delta^+$ is a second order Laplacian-type differential operator on the poloidal magnetic flux $\psi$, which in the large aspect ratio tokamak limit becomes the radial part of a cylindrical $\nabla^2$ operator.

Substituting the expression for $\langle \mu_0 \mathbf{J} \cdot \mathbf{B} \rangle$ from (56) in (51) and dividing by $\langle \mathbf{B} \cdot \nabla \zeta \rangle$, one finally obtains the desired parallel Ohm’s law in the form needed for the poloidal flux evolution equation (36):

$$\frac{\langle \mathbf{E} \cdot \mathbf{B} \rangle}{\langle \mathbf{B} \cdot \nabla \zeta \rangle} = \left(\frac{\eta_{nc}}{\mu_0} + \frac{\mu_e}{\nu_e} \eta_0 \right) \frac{1}{(R^{-2})} \frac{dP}{d\psi}.$$

(58)

Here, the first term indicates magnetic flux diffusion (see next section) induced by the neoclassical parallel resistivity, the second is due to electron inertia, and the last is caused by the neoclassical bootstrap current which for $\nu_{ec} = 0$ becomes simply $1.5(f_e/\nu_e)\eta_0(R_0/B_p)(dP/d\psi) \sim 1.1\sqrt{\epsilon} (d\beta_p/\epsilon)(\eta_0/\mu_0)(B_p/R_0)$ in the large aspect ratio tokamak limit. If an extra, non-inductive force on electrons $\mathbf{F}_e$ is added to the electron momentum equation (37), it would induce an additional “source” current $(\mathbf{J}_S \cdot \mathbf{B}) \equiv (\mathbf{F}_e \cdot \mathbf{B})/(\mu_0 \eta_{nc})$, and hence an extra contribution of $-\eta_{nc} (\mathbf{J}_S \cdot \mathbf{B})/(\mathbf{B} \cdot \nabla \zeta)$ on the right of (58).

V. MAGNETIC FLUX, FIELD LINE DIFFUSION

Magnetic field lines are definable and do not change their topology in ideal MHD. Specifically, using a Clebsch representation $\mathbf{B} = \nabla \alpha \times \nabla \beta$, one first notes that the local direction of $\mathbf{B}$ is given by the cross product of the gradients of $\alpha$ and $\beta$, which are perpendicular to the $\alpha$ and $\beta$ surfaces. Hence, a magnetic field line lies at the intersection of $\alpha = \text{constant}_1$ and $\beta = \text{constant}_2$ surfaces. Next, using the Clebsch representation and the ideal MHD Ohm’s law $\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$ in Faraday’s law (25), one can show that

$$d\alpha/dt = 0, \ d\beta/dt = 0, \ \text{with} \ d\alpha/dt \equiv \partial/\partial t + \mathbf{V} \cdot \nabla. \quad (59)$$

Since $\alpha$ and $\beta$ do not change in time on a given magnetic field line, even as it moves, they provide labels to define it in the plane perpendicular to $\mathbf{B}$ in ideal MHD. A convenient third coordinate in the Clebsch representation is the length $\ell$ along a magnetic field line which is determined from the projection of the equation for a magnetic field line ($d\ell/dt = \mathbf{B}/B_0$) in a desired direction.

For the axisymmetric magnetic field model in (3), one can identify the Clebsch coordinates as $\alpha \rightarrow \psi$ and $\beta \rightarrow q \theta - \zeta$; the length $\ell$ along magnetic field lines is determined from (15). However, there is a well-known problem with Clebsch coordinates on irrational magnetic field lines: when $q$ is not the ratio of integers the field line label $\beta = q \theta - \zeta$ is not periodic in $\theta$.

This periodicity difficulty will be addressed by introducing a local coordinate system in the vicinity of a given rational surface at $\rho = \rho_\circ$ where $q_{\circ} \equiv q(\rho_\circ) = m/n$. Namely, a new angular coordinate is introduced [16]:

$$\alpha \equiv \zeta - q_{\circ} \theta = \zeta - (m/n) \theta, \ \text{helical angle}. \quad (60)$$
Note that $\nabla \alpha$ is perpendicular (at all $\rho$) to the helical trajectory of the $q_\rho$ rational field line. Since $\nabla \theta \times q_\rho \nabla \theta = 0$, the Jacobian $\sqrt{g} \equiv (\nabla \rho \cdot \nabla \theta \times \nabla \alpha)^{-1}$ for transforming from the fixed Eulerian grid $x$ to the coordinates $\rho, \theta, \alpha$ is the same as before and thus given by (7). Writing $B$ in (3) as $B = \nabla \psi \times (q_\rho \theta - \xi) + (q - q_\rho)\theta$, it can be rewritten in terms of these new coordinates as a helical component with the pitch of the rational field line ($B_h$) plus a magnetic-shear-induced component ($B_s$):

$$B = \nabla \alpha \times \nabla \psi + \nabla \psi \times \nabla \theta \equiv B_h + B_s = \nabla \times A,$$ for $A \equiv -\psi \nabla \alpha + \psi_\ast \nabla \theta$. (61)

Here, the helical magnetic flux $\psi_\ast$ is defined by

$$\frac{\partial \psi_\ast}{\partial \rho} = (q - q_\rho) \frac{\partial \psi}{\partial \rho}, \quad \text{helical flux definition.} \quad (62)$$

It can be integrated across a rational surface to yield

$$\psi_\ast(\rho, t) = \int_{\rho_s}^{\rho} \frac{\partial \psi_\ast}{\partial \rho}(q - q_\rho) = \int_{\psi_{\ast}(\rho_s)}^{\psi_{\ast}(\rho)} d\psi_\ast - q_\ast \int_{\psi_{\ast}(\rho_s)}^{\psi_{\ast}(\rho)} d\psi.$$ (63)

Using the Taylor series expansion of $q(\rho)$ about the rational surface in (18), one obtains from (63)

$$\psi_\ast \approx \left(x^2 / 2\right) q' \psi' + O(x^3), \quad \text{helical flux.} \quad (64)$$

Here, $q'$ is as defined in (21) and $\psi' \equiv \partial \psi/\partial \rho|_{\rho_s}$. The helical flux function can also be defined by the relation $\psi_\ast \equiv \int d\alpha(B) \cdot A = \int_{\alpha_s}^{\alpha} d\alpha \nabla \alpha \cdot B$.

The introduction of plasma resistivity leads to radial diffusion of magnetic field lines. In Section III it was shown that while the toroidal magnetic flux $\psi_t$ does not change in time [in the Lagrangian frame moving with the grid velocity $u_g$, defined in (30)], the poloidal magnetic flux does change if there is an electric field component parallel to the magnetic field $B$. Substituting the Ohm’s-law-determined parallel electric field in (58) into the poloidal flux evolution equation (36), one obtains the diffusion-type (at least for $\delta_e^2 \Delta^+ << 1$) equation

$$\frac{d}{dt} \left(1 - \delta_e^2 \Delta^+\right) \psi = D_\eta \Delta^+ \psi - S_\psi,$$ (65)

in which the diffusivity for the poloidal magnetic flux (and hence for poloidal magnetic field lines) is

$$D_\eta \equiv \eta_{\|}/\mu_0, \quad \text{magnetic field diffusivity,} \quad (66)$$

and the source $S_\psi$ of poloidal magnetic flux is

$$S_\psi \equiv \frac{\partial \psi}{\partial t} - \frac{\mu_e}{\nu_e} \eta_0 \frac{1}{(R^{-2})} \frac{dP}{d\psi}.$$ (67)

The two sources of poloidal flux are due to the “current-drive” effects of the magnetic flux change in the central solenoid (first term: $\partial \psi/\partial t \equiv V^c_{\text{loop}} / 2\pi$) and the bootstrap current (second term); both contribute positively to $S_\psi$ for the usual confined plasma situation where $dP/d\psi < 0$. An extra, noninductive current source $J_S$ would add a source term $\eta_{\|}^2 (J_S \cdot B)/I(R^{-2})$ to (67).

For scale lengths longer than the em skin depth $\delta_e$ (i.e., for $\delta_e^2 \Delta^+ << 1$), the poloidal flux evolution equation (65) becomes a diffusion (parabolic) equation for $\psi$. Comparing this diffusion equation with (29), one sees that the poloidal flux $\psi_t$ and hence the poloidal magnetic field lines or bundles of them, diffuse radially (relative to the toroidal flux $\psi_t$ and hence toroidal field lines). Note from the definition of $\Delta^+ \psi$ in (56) that a necessary condition for diffusion of poloidal flux is that there is a flux-surface-averaged parallel current $(J \cdot B)$ in the plasma.

Consider first the equilibrium solution of (65) for the poloidal magnetic flux $\psi$. For equilibrium in the Lagrangian frame, $d/\partial t \to 0$ and the equation for the stationary poloidal flux $\psi_0$ becomes

$$0 = D_\eta \Delta^+ \psi_0 - S_\psi, \quad \psi \text{ equilibrium.} \quad (68)$$

Thus, in equilibrium the diffusion of $\psi$ (and hence of poloidal field lines) is balanced by the source $S_\psi$ of the poloidal magnetic flux, field lines: The Poynting flux represented by $\partial \psi/\partial t$ brings field lines into the plasma and the magnetic field diffusivity $D_\eta$ diffuses them out of the plasma — even for a stationary poloidal magnetic field.

This situation of the diffusion of a quantity balancing a source of it is typical of any diffusive process in stationary equilibrium. For example, the density evolution equation with a density source $S_n$ and a Fick’s diffusion law $\Gamma \equiv n \nabla V = -D \nabla n$ becomes $\partial n/\partial t = D \nabla^2 n + S_n$; in equilibrium this yields $0 = D \nabla^2 n + S_n$. (The sign of the density source $S_n$ is opposite to $S_\psi$ because while $\psi$ increases with radius, $n$ usually decreases with radius; hence, $\Delta^+ \psi > 0$ while $\nabla^2 n < 0$.) The Fick’s law radial diffusive “flux” of poloidal magnetic flux $\psi$ is $\Gamma_\psi \sim -D_\eta \psi \partial \psi/\partial r$ in a large aspect ratio tokamak with uniform resistivity.

Physically, an initially localized poloidal flux spreads radially in time via diffusion. A mathematical description of the diffusive spreading on short time scales can be developed by writing the poloidal flux in equilibrium ($\psi_0$) and localized transient $[\psi(x, t)$ for which $a^2 \Delta^+ \psi \approx \partial^2 \psi/\partial x^2 >> \delta^2 \psi$] parts: $\psi \approx \psi_0 + \delta \psi$. Substituting this Ansatz into (65) and taking account of the equilibrium equation (68), one obtains (for $x^2 \ll 1$)

$$\left(\frac{\partial}{\partial t} + \bar{u}_g \frac{\partial}{\partial x} \right) \left(1 - \delta_e^2 \frac{\partial^2}{\partial x^2}\right) \delta \psi \approx \bar{v}_e \delta^2 \frac{\partial^2 \delta \psi}{\partial x^2}, \quad (69)$$

in which the following normalized variables have been introduced:

$$\bar{u}_g \equiv \langle u_g \cdot \nabla \rho \rangle, \quad \text{normalized grid velocity (s$^{-1}$)}, \quad (70)$$
$$\delta_e \equiv \delta_e / \bar{a}, \quad \text{normalized em skin depth (<< 1)}, \quad (71)$$
$$\frac{1}{\bar{a}^2} \equiv \frac{1}{\langle R^{-2} \rangle} \left(\frac{\nabla \rho^2}{R^2} \right) \sim \frac{1}{a^2}, \quad \text{effective radius}, \quad (72)$$
$$D_\eta \equiv D_\eta / \bar{a}^2, \quad \text{normalized diffusivity (s$^{-1}$)}, \quad (73)$$
$$\bar{v}_e \equiv \nu_e (\eta_{\|}^2 / \eta_0), \quad \text{effective neoclassical} \nu_e. \quad (74)$$
Properties of solutions of (69) on the $1/\nu_e$ time scale can be explored by taking a Fourier transform in the $x$ (radial) variable and neglecting (for now) the grid velocity term $\bar{u}_g \partial / \partial x$, to yield

$$\frac{\partial \delta \hat{\psi}}{\partial t} = -\nu_e \frac{k_e^2 \delta_e^2}{1 + k_e^2 \delta_e^2} \delta \hat{\psi}. \tag{75}$$

Here, $\delta \hat{\psi}(k_e, t)$ is the Fourier transform of $\delta \psi(x, t)$. On radial scale lengths shorter than the em skin depth ($|x|^2 < k_e^2 \delta_e^2, k_e^2 \delta_e^2 > 1$), (75) yields $\delta \hat{\psi} / \partial t \simeq -\nu_e \delta \hat{\psi}$, whose solution $\delta \hat{\psi} = \delta \hat{\psi}_{|t=0}(1 - e^{-\nu_e t})$ approaches a constant on the transport time scale ($t > 1/\nu_e$). Physically, the poloidal magnetic field within the skin depth layer is spatially and temporally constant on the transport scale because magnetic field lines are not diffusing there; instead, since this is a “reactive” region because the electrical resistivity is imaginary there, the poloidal magnetic field associated with $\delta \hat{\psi}$, $\delta \mathbf{B}_p \equiv \nabla \zeta \times \nabla \hat{\psi}$, vanishes in this region ($|x|^2 < \delta_e^2$).

For radial scale lengths longer than the em skin depth $\delta_e$ (for which $k_e^2 \delta_e^2 < 1$ so that $\delta_e^2 \partial / \partial x^2$ can be neglected compared to unity), the evolution equation (69) for a localized poloidal magnetic field $\delta \psi$ becomes

$$\frac{d \delta \psi}{dt} \equiv \left( \frac{\partial}{\partial t} + \bar{u}_g \frac{\partial}{\partial x} \right) \delta \psi = D_q \frac{\partial^2 \delta \psi}{\partial x^2}. \tag{76}$$

The Green-function-type solution of this diffusion equation for a $\delta \psi$ initially radially localized at $x = x_0$ ($> \delta_e$), represented by $\delta \psi(x, t = 0) = \delta \psi_0 \delta(x - x_0)$, is (for $t \geq 0$)

$$\delta \psi(x, t) = \delta \psi_0 e^{-\frac{(x-x_0-\bar{u}_g t)^2}{4D_q t}}. \tag{77}$$

In view of the discussion in the preceding paragraphs, this solution is valid for $1 > x^2 > \delta_e^2$. Note that $\delta \psi$ provides a temporally evolving probability distribution for the radial location of the poloidal flux $\delta \psi_0$ initially at $x = x_0$ (i.e., at $\rho = \rho_e + x_0$, which represents an arbitrary radial position for the present poloidal flux discussion) as it moves (advepts) radially with the grid velocity $\bar{u}_g$ and diffuses radially due to $D_q$ in a time $t$.

The average radial displacement and spread (variance) of the initially localized flux grow linearly with time:

$$\frac{\int_{-\infty}^{\infty} dx (x - x_0) \delta \psi}{\int_{-\infty}^{\infty} dx \delta \psi} = \bar{u}_g t, \tag{78}$$

$$\frac{\int_{-\infty}^{\infty} dx (x - x_0)^2 \delta \psi}{\int_{-\infty}^{\infty} dx \delta \psi} = 2 D_q t = (2 \nu_e t) \delta_e^2. \tag{79}$$

Thus, poloidal magnetic field lines advect radially at speed $\bar{u}_g$ and diffuse radially a root mean square distance $x_{rms} = \delta_e(2 \nu_e t)^{1/2}$ in a time $t$.

This poloidal flux (field line) diffusion process is analogous to how a group of “marked particles” would diffusively spread away from some positied initial position in the density evolution equation analogy discussed after (68). In the paleoclassical model the “marked particles” will be the small amount of poloidal magnetic flux traversed by an electron gyroorbit and the locally Maxwellianized distributions of electrons and hence equilibrated electron temperatures on these field lines. In the next section a Fokker-Planck formalism will be developed for including the effects of the radial advection and diffusion of poloidal flux on the electron guiding centers and hence on the electron distribution function in a kinetic analysis. Note that this advection and diffusion process occurs even when the poloidal magnetic field is in stationary equilibrium (i.e., $d\psi / dt = 0$), just as marked particles diffuse away from an initial position in the equilibrium density analogy discussed between (68) and (69).

Implicitly, the preceding analysis assumed the poloidal magnetic flux is in stationary equilibrium (at least in a Lagrangian frame moving with the grid velocity $\bar{u}_g$), i.e., $d\psi / dt = 0$, as used to obtain (68). When the poloidal flux is not stationary, one can define (neglecting $\bar{u}_g$)

$$\psi \equiv \partial \psi / \partial t = D_q \Delta^+ \psi - S_\psi. \tag{80}$$

By analogy with identifying $\partial \Psi / \partial t$ [after (34)] as the externally applied loop voltage, $\psi$ is the local change in the “Lenz’s law” voltage on a given flux surface in the plasma: $2\pi \psi = \delta \mathcal{V} \zeta (\rho, t) \equiv -\int d\ell (\zeta) \cdot \delta \mathbf{E}$ with $\delta \mathbf{E} \equiv -\partial \delta \mathbf{A} / \partial t$ and $\delta \mathbf{A} = -\hat{\psi} t \nabla \zeta$. [The loop voltage is $-\int \delta \mathbf{E} \cdot \delta \mathbf{A}$ because $d\ell (\zeta)$ is in the $-\nabla \zeta$ direction.]

When $\psi \neq 0$, the poloidal flux is changing in the plasma. This causes the $x$ position of a given poloidal flux surface to move (relative to the toroidal flux $\psi_t$). To determine this motion, the poloidal magnetic flux $\psi(p, t)$ can be expanded in $x$ and $t$ (assuming the flux changes slowly, i.e., $\psi \ll \bar{u}_g \psi_0$):

$$\psi(x, t) \simeq \psi(0, 0) + x \psi' + t \psi' + \cdots. \tag{81}$$

Setting the total derivative of this expansion to zero [to remain on the same magnetic flux surface as $\psi$ changes slowly in time — see (59)], one obtains the radial speed of a field line induced by $\psi$:

$$\bar{u}_\psi = \frac{dx}{dt} \equiv -\frac{\psi'}{\psi} = -\frac{\partial \psi / \partial t}{\partial \psi / \partial \rho}, \quad u_\psi = \bar{u}_\psi \hat{\psi}. \tag{82}$$

In the next section the effects of the motion of poloidal magnetic flux will be characterized (in a kinetic theory context) by a Fokker-Planck [17, 18] description. The relevant Fokker-Planck coefficients for the radial advection and diffusion of the poloidal flux can be defined by

$$\frac{\langle \Delta x \rangle}{\Delta t} \equiv \int_{-\infty}^{\infty} dx (x - x_0) W(x_0; x),$$

$$\frac{\langle (\Delta x)^2 \rangle}{\Delta t} \equiv \int_{-\infty}^{\infty} dx (x - x_0)^2 W(x_0; x),$$

in which the “transition probability” $W(x_0; x)$ is the probability per unit time that a poloidal field line (infinitesimal amount of poloidal flux) will move from $x_0$.
to \(x\), for small departures of \(x\) from \(x_0\): \(|x - x_0| << 1\).

Note that angular brackets in Fokker-Planck theory indicate ensemble average or “expectation” values — not the magnetic flux surface average in (10).

For the poloidal flux (field line) evolution, the appropriate transition probability is given by \(W(x_0; x) = \partial[\delta\psi(x, t)]/\partial x_0\). Thus, adding the effects in the preceding paragraphs together, the motion and diffusive spread of the poloidal flux, field lines can be written in terms of Fokker-Planck coefficients as

\[
\frac{(\Delta x\psi)}{\Delta t} = \bar{u}_\psi + \bar{u}_\psi, \quad \left(\frac{(\Delta x\psi)^2}{\Delta t}\right) = 2D_\eta. \tag{83}
\]

These results are applicable only for \(|x| > \tilde{\delta}_\epsilon\); in the opposite limit the coefficients vanish.

In the next section the effects of magnetic field diffusion will be considered in the vicinity of a rational surface. Thus, the motion and diffusion of the local helical flux (field lines) needs to be considered. Physically, since from (63) the helical flux depends on the toroidal flux \(\psi_t\), which does not move (in the Lagrangian frame) and the poloidal flux \(\psi\), which does advect and diffuse, one can anticipate that the localized helical flux will mostly advect and diffuse just like the poloidal flux does.

To explore the behavior of the helical flux, one begins by taking the time derivative of the definition of \(\psi\) in (63) using \(d\psi/dt = 0\) from (29) and (65), to obtain (neglecting skin depth effects for \(x^2 > \delta^2\)) by an analysis similar to the discussion in the paragraph following (75)

\[
d\psi_*/dt = -q_*d\psi/dt = -q_* (D_\eta\Delta^+\psi - S_\psi). \tag{84}
\]

Next, the last definition of \(\psi_*/(63)\) is used to write

\[
\Delta^+\psi = \Delta^+(-\psi_* + \psi_t)/q_* \tag{85}
\]

Close to a rational surface the local radial coordinate \(x\) defined in (20) is small and the \(\Delta^+\) differential operators on \(\psi_*\) and \(\psi_t\) can be approximated by \(\Delta^+\psi_* \simeq (1/\alpha^2)(\delta^2\psi_*/\delta \alpha^2)\) and \(\Delta^+\psi_t \simeq q_*\Delta^+\psi + q^*\psi'/\alpha^2\), which was obtained using the Taylor series expansion \(d\psi_*/d\rho = q\partial\psi/\partial \rho \simeq (g + xq' + \cdots)\partial\psi/\partial \rho\). Using these results and writing the helical flux in terms of equilibrium and localized transient parts \(\psi_* = \psi_{*0} + \delta\psi_*(x, t)\) where from (63) \(\psi_{*0} \simeq (x^2/2)q^*\psi'/\alpha^2\) so that \(\Delta^+\psi_{*0} \simeq q^*\psi'/\alpha^2\), one obtains from (84) and (85) for \(\delta^2 < x^2 << 1\)

\[
\left(\frac{\partial}{\partial t} + \bar{u}_\psi \frac{\partial}{\partial x}\right) \delta\psi_* \simeq \bar{D}_\eta \partial^2 \delta\psi_*/\partial x^2 - q_* \bar{\psi}. \tag{86}
\]

The differential operators and Green-function-type solution for \(\delta\psi_*\) are the same as those in (76) and (77).

Thus, helical magnetic field lines with \(q_* \equiv m/n\) also advect with the velocity \(\bar{u}_\psi\) and diffusively spread with diffusivity \(\bar{D}_\eta\), as indicated in (78) and (79).

When the poloidal magnetic field is not in stationary equilibrium (i.e., \(\psi \neq 0\)), the radial position of the rational surface changes. Setting the time derivative of the \(q\) in (5) to zero one finds the rational surface motion induced by \(\bar{\psi} = \rho_* (\partial\psi/\partial \rho)/(\partial\psi/\partial \rho))\). In addition, the helical flux changes according to \(d\delta\psi_*/dt = -q_*\bar{\psi}\). Expanding such an evolving helical flux about a (moving) rational surface as in (81), one obtains \(\psi_*(x, t) \simeq \psi_*(0, 0) + (x^2/2)q^*\psi'/t - q_*\bar{\psi} + \cdots\). Setting the total derivative of this helical flux function to zero (to find the motion required to stay on the same helical field line), one finds [analogously to (82)] \(d(x^2/2)/dt = q_*\psi'/q^*\psi'\).

Unlike the advective velocity \(\bar{u}_\psi\) of poloidal field lines induced by \(\psi \neq 0\), for helical field lines \(\psi \neq 0\) induces a quadratic spreading (for \(\psi > 0\)) or contraction (\(\psi < 0\)) motion — because \(\psi_* \sim x^2\). However, it is not a diffusive (i.e., Markov [17, 18]) process. Since these effects are somewhat complicated, for simplicity they will be circumvented here by assuming the induced transients are so weak that the rational surface moves less than the \(\psi\) surface and the diffusion of helical field lines dominates over the \(\psi\)-induced spreading, contraction:

\[
\psi >> \rho_* \frac{\partial\psi_*/\partial \rho}{\rho_*}, \quad \bar{D}_\eta >> \frac{q_* \psi_*/ \partial \psi'/\rho_*}. \tag{87}
\]

Physically, these criteria imply the poloidal flux change must be nearly homogeneous in the plasma and the induced parallel inductive electric field \((\partial E \cdot B)\) must be smaller than the equilibrium resistive electric field induced by the parallel current in the plasma, \(\eta^{ne}\) \((J \cdot B)\).

Some final notes are in order when magnetic fluxes (field lines) are diffusing. While in ideal MHD magnetic field lines can be assigned unique labels (i.e., Clebsch coordinates \(\alpha\) and \(\beta\), this is no longer possible when plasma resistivity effects are added — because as an initially localized magnetic flux diffuses radially according to (76) or (86), it assumes a probability distribution [see (77)] centered about a radial position determined by the advecting radial grid velocity, but with a radial spread (variance) of order \(\delta_e \sqrt{\eta_\rho}\) that increases with time \(t\). Note also that the diffusing poloidal magnetic flux does not induce any “magnetic flutter” [19] (i.e., \(\mathbf{B} \cdot \nabla \psi \neq 0\)) — because the advecting and diffusing field lines always lie on flux surfaces and the solenoidal condition \(\nabla \cdot \mathbf{B} = 0\) is satisfied automatically at all times by the flux function representations of \(\mathbf{B}\) in (3) and (61). Since plasma resistivity causes poloidal magnetic flux (field lines) to diffuse radially, individual field lines lose the uniqueness they have in ideal MHD. Thus, it is better to think of \(\psi\) (and \(\psi_t, \psi_*, x\)) as the radial coordinate of a particular magnetic flux surface or of magnetic field lines on that surface — and realize that they are not invariant labels for particular field lines as they diffuse radially. In most plasma physics studies the fact that magnetic flux is diffusing radially can be neglected — because the flux surface labels (\(\psi, \psi_t, \psi_*\) or \(x\)) remain unchanged. However, poloidal magnetic flux diffusion is important in axisymmetric toroidal plasmas through its effects on an electron’s guiding center position.
VI. PALEOCLASSICAL KINETICS, ANALYSIS

Paleoclassical radial transport will be caused by electrons and their heat being nearly “frozen to” and hence carried with the poloidal magnetic flux $\psi$ as it advects and diffuses radially. The Lorentz force $q_e v \times B$ causes electrons to gyrate rapidly (at the electron gyrofrequency $\omega_{ce} \equiv q_e B/m_e$) with typical electron gyroradii $q_e \equiv v_{Te}/\omega_{ce} \lesssim 0.1$ mm $<< \delta_e \sim 1$ mm around a magnetic field line. In the absence of magnetic field diffusion, a key constant of the motion for axisymmetric systems is the canonical toroidal angular momentum $p_\zeta \equiv R^2 \nabla \zeta \cdot (m_e v + q_e A) = m_e R^2 \nabla \zeta \cdot v - q_e \psi$ in which $A = \psi \nabla \theta - \psi \nabla \zeta$ from (3) has been used. Since to lowest order in a small gyroradius expansion $p_\zeta \sim -q_e \psi$, one usually thinks of electrons as being “stuck to” the poloidal flux $\psi$ at their guiding center, or at least to the small amount of poloidal magnetic flux traversed by their radial gyromotion. The lowest order, “collisionless” differential equation for simplicity):

$$m_e \frac{dv}{dt} = q_e \left(-\frac{\partial A}{\partial t} + v \times B\right). \quad (88)$$

Taking the projection of this equation in the $e_\zeta = R^2 \nabla \zeta$ (covariant base vector) direction, one obtains

$$m_e \frac{d(R^2 \nabla \zeta \cdot v)}{dt} = q_e \left(\frac{\partial \psi}{\partial t} + v \cdot \nabla \psi\right) = q_e \frac{d\psi}{dt}, \quad (89)$$

which is valid on the gyromotion and longer time scales. (The same equation is applicable for the local helical geometry since the covariant base vector $e_\zeta = R^2 \nabla \zeta = e_\zeta$.)

Usually one simply integrates this equation over time to obtain the $p_\zeta = m_e R^2 \nabla \zeta \cdot v - q_e \psi$ constant of electron motion. However, the previous section showed that the poloidal magnetic flux $\psi$ obeys a diffusion equation — from (65) $d\psi/dt = D_\psi \nabla^2 \psi - S_\psi$ for $x^2 > \delta_e^2$. Because the poloidal magnetic field energy is being continuously dissipated (i.e., it is a non-conservative field), Newton’s second law for our system cannot be transformed into an area-preserving set of Hamiltonian-type equations — at least not on the poloidal field dissipation time scale.

The effect of the poloidal flux advection and diffusion will be taken into account via a multiple-time-scale analysis. First, one notes that on the gyromotion time scale $p_\zeta$ is a constant of the motion and the radial ($\psi_1, \rho$ or $x = \rho - \rho_*$) position of the electron is oscillatory. Thus, to first order in a small gyroradius expansion one writes

$$\psi(x,t) = \psi(x_0) + \delta \psi \sin(-\omega_{ce} t + \varphi_0), \quad (90)$$

in which $x_0$ is the radial position of the electron’s guiding center and $\delta \psi \sim q_e \partial \psi / \partial \rho << \psi_0$ is the small amount of poloidal magnetic flux traversed by the radial component of the electron gyromotion.

Normally one thinks of $\delta \psi$ as an invariant type quantity determined by the electron gyroadius. However, when $\eta (\mathbf{J} \cdot \mathbf{B}) \neq 0$, the small, radially localized flux $\delta \psi$ advects and diffuses radially — as indicated by the Fokker-Planck coefficients given in (83). This effect is small and accumulates to become significant only on a much longer time scale than the electron gyroperiod $\tau \equiv 2\pi / \omega_{ce}$; it will be taken into account as a net small effect (mapping) per electron gyroperiod. Since the radial $x$ excursion of the electron is determined by $\delta \psi(x,t)$, over one electron gyroperiod the gyromotion-traversed flux $\delta \psi$ and hence the electron’s guiding center radial position $x_g$ advects and diffuses radially by

$$\langle \Delta x_g \rangle = \left< \left( u_g + \hat{u}_g \right) \tau \right>, \quad \langle (\Delta x_g)^2 \rangle = 2 \hat{\Delta}_g \tau. \quad (91)$$

This radial diffusion per gyroperiod is quite small:

$$a^2 (\langle (\Delta x_g)^2 \rangle / \delta_\tau) \sim 4 \pi \left( \nu_e / \omega_{ce} \right) (\delta_e^2 / \delta_\tau^2) \sim 10^{-3} \text{ for a typical case with } \nu_e \sim 10^5 / \text{s and } \omega_{ce} \sim 10^{11} / \text{s. However, let one notes that it is negligibly small, note that on the same time scale Coulomb collisions induce velocity-space diffusion and hence diffusion of the electron gyroradius, which is the physical origin of classical transport, of only } \langle \Delta v \psi \rangle \rangle / \nu^2 \sim \nu_e \tau \sim \omega_{ce} \langle (\Delta n \lambda^2_D) / \nu^2 \rangle \sim 10^{-6} \text{ for a typical case with } \omega_{ce} \sim 10^{11} / \text{s.}$$

Taking the electron gyromotion period $\tau$ to be the relevant stochastic time step $\Delta t$, the net result of the preceding analysis is that electron guiding centers advect and diffuse radially with the same Fokker-Planck coefficients as those for poloidal magnetic field lines, i.e., those given in (83). These effects will be included in a gyro-averaged kinetic equation by adding spatial Fokker-Planck operators that represent the electron guiding center motion due to magnetic flux advection and diffusion effects — analogous to how the velocity-space diffusion effects of Coulomb collisions were originally added [21] to the “collisionless” Vlasov equation. Because some readers may not regard the preceding arguments for the Fokker-Planck coefficients for electron guiding center advection and diffusion to be sufficiently rigorous, this paper will take them to be the key hypothesis of the paleoclassical model and proceed assuming they are valid.

The appropriate electron kinetic equation is the gyro-averaged one, which is called the drift-kinetic equation [20]. Adding Fokker-Planck-type effects [17, 18] of guiding center motion due to magnetic flux advection and diffusion, the magnetic-field-diffusion-Modified Drift-Kinetic Equation (MDKE) becomes

$$\frac{\partial f}{\partial t} + \mathbf{v}_g \cdot \nabla f + \hat{\mathbf{v}} \cdot \frac{\partial f}{\partial \mathbf{v}} = \mathcal{C} \{ f \} + \mathcal{D} \{ f \}. \quad (92)$$
Here, \( f = f(x_g, \varepsilon, \mu, t) \) is the guiding center distribution function, \( x_g \) is the guiding center position (\( \sim x \) because the electron gyroradius is negligibly small), the magnetic moment \( \mu \equiv m_v \mu_B \), and kinetic energy \( \varepsilon \equiv m_v^2/2 = m_v^2/2 + \mu B \). Further, the guiding center velocity \( v_g = v_0 B/B + v_D \) in which \( v_0 = B \cdot \nabla f/|B| = \pm \sqrt{2/m_e} \varepsilon \) (\( \varepsilon - \mu B \)) is the parallel (to \( B \)) component of the electron velocity \( v \) and \( v_D \) is the electron drift velocity, \( \varepsilon \approx \mu B \partial B/\partial t + q_e v_g \). \( \varepsilon \approx q_e v \cdot E \) is solely a function of the poloidal \( E \) field, \( \varepsilon \approx q_e v \cdot E \cdot B/B, \) and \( C\{f\} \) is the Coulomb collision operator.

Finally, effects due to magnetic flux (field line) advection and diffusion are indicated by the Fokker-Planck spatial diffusion operator \( D \). Because of the complexity of the magnetic geometry, the Fokker-Planck coefficients must be written in a general vectorial form, which is in terms of the covariant base vector in the “radial” direction \( e_r = \partial x/\partial \rho = \sqrt{g} \nabla \theta \times \nabla \zeta, \) for which \( e_r \cdot \nabla \rho = 1 \):

\[
\frac{\langle \Delta x \rangle}{\Delta t} = \frac{\langle \Delta x \rangle}{\Delta t} e_r, \quad \frac{\langle \Delta x \Delta x \rangle}{\Delta t} = \frac{\langle \Delta x \rangle^2}{\Delta t} e_r e_r. \tag{93}
\]

Because in the paleoclassical model the electron guiding centers advect and diffuse with the poloidal field lines, the Fokker-Planck coefficients \( \langle \Delta x \rangle/\Delta t \) and \( \langle \Delta x \Delta x \rangle/\Delta t \) are those specified in (83). In full three-dimensional geometry the Fokker-Planck differential operator is [17, 18]

\[
D\{f\} \equiv -\nabla \cdot \langle \Delta x \Delta x \rangle/\Delta t f + \frac{1}{2} \nabla \cdot \langle \Delta x \Delta x \Delta x \rangle/\Delta t f. \tag{94}
\]

Using (12), the definition of the divergence of a vector field, \( \nabla \cdot A = \sum_i (1/\sqrt{g})(\partial A^i/\partial u^i)(\sqrt{g} \nabla \theta \times \nabla \zeta) \), and neglecting \( \nabla \rho \cdot \partial e_r/\partial \rho = \langle \nabla \rho \cdot \partial x/\partial \rho \rangle \sim O(\varepsilon^2) \), when the distribution function \( f \) is solely a function of a magnetic flux coordinate (i.e., \( \rho, \psi \) or \( \psi_e \)), the flux-surface-average of this operator becomes

\[
D\{f(\rho)\} \approx \frac{1}{V} \frac{\partial}{\partial \rho} \left( -V^r \langle \Delta x \Delta x \rangle/\Delta t f + \frac{\partial}{\partial \rho} V^r \langle \Delta x \Delta x \rangle/\Delta t f \right) = \frac{1}{V} \frac{\partial}{\partial \rho} \left( -V^r (\bar{u}_r \bar{u}_g) f + \frac{\partial}{\partial \rho} V^r D_n f \right). \tag{95}
\]

\begin{align*}
\text{A. Axisymmetric Paleoclassical Transport}
\end{align*}

The transport induced by adding magnetic field line diffusion effects to the drift-kinetic equation will be explored first for a toroidally axisymmetric distribution function (i.e., \( \partial f/\partial \xi = 0 \)). Then, using the usual magnetic field model in (3), (92) can be simplified to

\[
\frac{\partial f}{\partial t} + \frac{v || |(B \cdot \nabla \theta)| \partial f}{|B|} + v_D \cdot \nabla f + \frac{\xi}{\varepsilon} \frac{\partial f}{\partial \varepsilon} = C\{f\} + D\{f\}. \tag{96}
\]

Here, \( f = f_\sigma(\psi, \theta, \mu, \varepsilon, t) \), in which the subscript \( \sigma \) indicates the axisymmetric distribution, and from (6) and (7) \( B \cdot \nabla \theta = \psi \sqrt{g} \) is solely a function of the poloidal \( E \) field, \( B \cdot \nabla \theta = \psi \sqrt{g} \approx B_p/r = B_p/R_0 g \).

As usual, solutions of the kinetic equation will be sought using a small gyroradius ordering scheme in which

the first order terms are linear in the gyroradius, em skin depth \( \delta \), ratio of collision frequency to gyrofrequency, and ratio of the electric field to the Dreicer (runaway) electric field. The second order terms are quadratic in these quantities and include transport-time-scale temporal evolution effects. The lowest order axisymmetric drift-kinetic equation obtained from (96), which includes parallel streaming \( (v || |(B \cdot \nabla \theta)|/B \sim v_{Te}/R_0 g) \) and Coulomb collision frequency (\( \sim v_c \)) effects on the lowest order axisymmetric distribution function \( f_{20} \), is

\[
\frac{v || |(B \cdot \nabla \theta)|}{|B|} \frac{\partial f_{20}}{\partial \theta} = C\{f_{20}\}. \tag{97}
\]

Its solution (for low collisionality plasmas in which \( \lambda_e \equiv v_{Te}/v_c > \pi R_0 q \), the poloidal periodicity length) is [5]

\[
f_{20} = f_M(\psi, \varepsilon, t) \left( \frac{m_e}{2 \pi T_e(\psi, t)} \right)^{3/2} e^{-\varepsilon/T_e(\psi, t)}. \tag{98}
\]

Thus, the lowest order axisymmetric distribution is a Maxwellian that is a function of the poloidal flux function \( \psi \); hence, the electron temperature and density are constant on the \( \psi \) magnetic flux surfaces (i.e., along poloidal field lines). (Physically, the electron temperature is equilibrated along field lines by parallel electron heat conduction.) The parametric dependence of the electron density \( n_e \) and temperature \( T_e \) on the time \( t \) allows for temporal evolution of these quantities on the transport time scale, which is second order in the small gyroradius expansion.

The first order drift-kinetic equation is [5]

\[
\frac{v || |(B \cdot \nabla \theta)|}{|B|} \frac{\partial f_1}{\partial \theta} - C\{f_1\} = -v_D \cdot \nabla \psi \frac{\partial f_M}{\partial \psi} - \frac{ev || |E|}{T_e} f_M. \tag{99}
\]

Solutions of this equation yield [5, 6] the first order (in the small gyroradius expansion): equilibrium current induced by the parallel electric field [which leads to the Spitzer electrical resistivity (53)], neoclassical flows and heat flows within the magnetic flux surface induced by the \( \mathbf{E} \times \mathbf{B} \) and diamagnetic flows, and the closure relations for the parallel viscous stress in (42) and (45).

The net electron energy transport equation including both neoclassical and axisymmetric paleoclassical effects is obtained by taking the kinetic energy moment \( \int f \varepsilon \nabla \cdot (m_v^2/2) \) of the axisymmetric MDKE (96), flux-surface-averaging it using (10) [5], and approximating \( D\{f_{20}\} \approx D\{f_M(\psi)\} \) to yield

\[
\frac{3}{2} \frac{\partial}{\partial t} \left( n_e T_e \varepsilon^{5/3} \right) + \frac{\partial}{\partial \psi} \left( (\varepsilon^{5/3} + \frac{5}{2} T_e \Gamma_e^{5/3}) \cdot \nabla \psi \right) + \frac{\partial}{\partial \psi} \left( Q_{\varepsilon}^{5/3} \cdot \nabla \psi \right) = Q_{\varepsilon}. \tag{100}
\]

Here, since the collisional entropy is \( s_e \equiv (3/2) \ln p_e V_e^{5/3} \), the first term represents \( p_e \partial s_e/\partial t \) — the adiabatic compression changes [5] in the electron pressure \( p_e \equiv n_e T_e \) on a flux surface \( \left[ V \rightarrow V^e(\psi) \right] \) in the presence of moving
poloidal flux surfaces ($\psi \neq 0$). The rest of the terms represent entropy-producing processes: the neoclassical conductive ($q_{\text{nc}}^e$) and convective \([5/2]T_e \Gamma_{\text{nc}}^e\) electron heat fluxes, the total (advective plus diffusive) axisymmetric poloidal electron heat flux \(Q_{\text{ea}}^{\text{pc}}\), and the electron heating \(Q_e\) due to collisional effects (joule heating, electron viscosity, and collisions with ions). The diffusive part of the neoclassical conductive heat flux contribution to (100) (in the $u$ rest frame) is \((\partial/\partial V)(q_{\text{nc}}^e \cdot \nabla V) \approx -(1/V')(\partial/\partial p)(V'/|\nabla p|^2) n_e \chi^e \partial T_e / \partial p\), in which $\chi^e_{\text{nc}} \sim \mu_e \beta_{\text{pe}} \sim v_e g_e q_e^2 / \epsilon_i^{3/2}$ is the neoclassical electron heat diffusivity \([5, 6]\).

The axisymmetric poloidal total electron heat flux $Q_{\text{ea}}^{\text{pc}}$ is due to the energy moment of $D\{f_M\}$. It is composed of advective and diffusive parts:

\[
\langle Q_{\text{ea}}^{\text{pc}} \cdot \nabla V \rangle = V' \left( \tilde{u}_g + \tilde{u}_\phi \right) \frac{3}{2} n_e T_e - \frac{\partial}{\partial p} \left( V' \tilde{D}_e \frac{3}{2} n_e T_e \right). 
\]

(101)

Physically, the electron internal energy \((3/2) n_e T_e\) is advected and compressed by the grid velocity $u_g$ and the poloidal field line motion induced by $\psi \neq 0$, and diffused radially with the magnetic field diffusivity $D_e$. When the advective flow is measured in the rest frame of the toroidal flux surfaces (i.e., $u_g$), as neoclassical flows are, the $\tilde{u}_g$ contribution to (101) is removed.

The diffusive poloidal electron heat flux in (101) indicates an axisymmetric (script a) poloidal radial electron heat diffusivity of

\[
\chi_{\text{ea}}^{\text{pc}} = \frac{3}{2} D_e = \frac{3}{2} \frac{\partial}{\partial p} \left( V' \tilde{D}_e \frac{3}{2} n_e T_e \right). 
\]

(102)

Note that this paleoclassical electron heat diffusivity is usually larger than the neoclassical electron heat diffusivity \([\chi_{\text{ea}}^{\text{pc}} / \chi_{\text{nc}}^{\text{pc}} \sim (\tilde{u}_g^2 \delta_e) / (\nu_e g_e q_e^2 / \epsilon_i^{3/2}) \sim \epsilon_i^{3/2} / (\alpha e q_e^2) > 1]\), especially near the plasma edge where $T_e$ and $\beta_e \equiv n_e T_e / (B^2/2\mu_0)$ become quite small. Thus, even the axisymmetric paleoclassical effects of diffusing poloidal flux (field lines) are important for collisional radial electron heat transport in axisymmetric toroidal plasmas.

### B. Representations of Nonaxisymmetric $f$

Next, consider the Fourier expansion of the distribution function in the poloidal ($\theta$) and toroidal ($\zeta$) angles:

\[
f(\psi, \theta, \zeta) = \sum_{m,n} f_{mn}(\psi) e^{im\theta - in\zeta} = \sum_m f_{m0}(\psi) e^{im\theta} + \sum_{m,n \neq 0} f_{mn} e^{im\theta - in\zeta}. 
\]

(103)

The $n = 0$ contributions (first sum on the second line) represent the axisymmetric parts of the distribution function that have been calculated in the preceding subsection. On and in the vicinity of a rational surface where $q(\rho_*) = m/n$, the appropriate angular variables are $\theta$ and the helical angle $\alpha \equiv \zeta - q_* \theta = \zeta - (m/n) \theta$. The nonaxisymmetric ($n \neq 0$, subscript na) contributions to $f$ can be put into a form that isolates the helical and poloidal angle dependences by first changing the poloidal harmonic sum in (103) from $m$ to $m'$, next making use of the definition of the helical angle $\alpha$, and finally defining a new poloidal harmonic summation variable $\tilde{m} \equiv m' - m$:

\[
f_{\text{na}}(\psi, \theta, \alpha) = \sum_{n \neq 0} e^{-\imath n \alpha} \sum_{\tilde{m}} f_{m+n,n}(\psi) e^{\imath \tilde{m} \theta}. 
\]

(104)

In the following subsection the “helical” kinetics and paleoclassical electron heat transport in the vicinity of a single medium order ($n \sim 10$) rational surface will be considered. The basic philosophy will be that on and in the vicinity of a rational field line there is helical symmetry (or “quasi-helical symmetry”) in the toroidal system, and that in a low collisionality plasma the electron temperature will equilibrate over the long axial length of the helical field lines (if $\lambda_e > \lambda_x$). Local helical flux coordinates (in particular the local radial variable $x \equiv \rho - \rho_e$ and helical angle $\alpha \equiv \zeta - q_* \theta$) will be relevant because the electron temperature will be equilibrated along the axially-extended helical field line as it diffuses radially away from the rational surface — into regions with $q \neq q_*$, because of the shear in the magnetic field.

Developing a useful (i.e., one-dimensional) representation of the nonaxisymmetric distribution function for $n > 1$ is analogous to the development of ballooning mode theory \([22, 23]\) or the ballooning transformation \([24]\). The basic issue is: how does one maintain periodicity of the solutions in the poloidal ($\theta$) and helical ($\alpha$) angles as one moves radially away from a helically symmetric rational surface (i.e., to $x \neq 0$ in the sheared magnetic field structure). The procedure for doing this for large $n$ “flute-like” plasma responses that are extended long distances ($|\ell| >> \pi R$) along magnetic field lines, such as for paleoclassical processes, is to assume $q$ is locally approximately a linear function of $x$ (i.e, $q \approx q_*$ and $xq^f$) and use ballooning transform theory \([22-24]\). Subsequent paragraphs in this section develop a useful representation for $f_{\text{na}}$ following the procedure Lee and Van Dam \([23]\) used to develop a ballooning representation.

Neglecting particle drifts off of field lines because in axisymmetric toroidal magnetic fields their radial excursions ($\Delta r \sim \rho_e q / \epsilon_i^{1/2}$) are smaller (for $\beta_e < |q|^{1/2}$) than the magnetic field diffusion step length $\delta_e \equiv c/\omega_e$, and because their dominant effects have already been included in the axisymmetric neoclassical theory, the relevant nonaxisymmetric MDKE \((92)\) is

\[
\partial f / \partial t + (v || / B) \cdot \nabla f = \mathcal{L} \{ f \} + D \{ f \}. 
\]

(105)

In the vicinity of a given rational surface the magnetic field can be represented in terms of its helical and magnetic shear components as in (61) and the parallel-streaming differential operator in (105) becomes \([16]\):

\[
B \cdot \nabla f = (B \cdot \nabla \theta) \left[ \frac{\partial f}{\partial \theta} \mid \psi, \alpha \right] + (q - q_*) \frac{\partial f}{\partial \alpha} \mid \psi, \theta. 
\]

(106)
Thus, on and in the vicinity of the $q_* \equiv m/n$ rational surface the relevant distribution function becomes $f \to f(\psi_*, \theta, \alpha, \mu, \xi, t)$. Note again from the definition of the Jacobian in (7) that $B \cdot \nabla \theta = \psi' / \sqrt{g} \approx B_p / r = B_0 / R_0 g$.

Applying this parallel-streaming operator to the nonaxisymmetric distribution form given in (104) yields

$$\mathbf{B} \cdot \nabla f_{na} = (\mathbf{B} \cdot \nabla \theta) \sum_{n \neq 0} e^{-in\alpha_n} \sum_{\tilde{m}} e^{i\tilde{m} \theta} \times i [\tilde{m} - n(q - q_*)] f_{m+\tilde{m},n}(\psi). \quad (107)$$

Since the parallel-streaming term $(v_p / B) \cdot \nabla f_{na}$ is dominant in (105), it will cause the Fourier coefficients $f_{m+\tilde{m},n}$ to be quite small (i.e., negligible) unless its multiplier $\tilde{m} - n(q - q_*)$ is small. Near the $q_* \equiv m/n$ rational surface $q$ can be expanded in a Taylor series as given in (18) and the free-streaming coefficient becomes

$$\tilde{m} - n(q - q_*) \approx \tilde{m} - n(m/n + x q' - q_*) = \tilde{m} - n x q'. \quad (108)$$

This will be smallest and lead to the largest $f_{m+\tilde{m},n}$ for $\tilde{m} = 0$ and $|n x q'| << 1$. The resulting “helically resonant” Fourier coefficient (near $q = q_*$) will be

$$f_*(x) \equiv f_{m,n}(\psi_*), \quad \text{helical kinetic distribution.} \quad (109)$$

Here, the $*$ and $m, n$ subscripts indicate the Fourier coefficients for the rational surface $q_* \equiv m/n$ and the argument has been changed from the poloidal $(\psi)$ to helical $(\psi_*)$ flux, which is the appropriate (radial) flux label near the given rational surface. Because from (22) the spacing between rational surfaces with different $m$ but the same $n$ is $\Delta \approx 1/nq'$, the criterion $|n x q'| << 1$ is just $|x| << \Delta$. Hence, the $f_*(x)$ solutions will be highly peaked near the $q_*$ rational surface; in particular, they will be radially localized to a region much narrower than the spacing between same rational surfaces.

Since the coefficients in (105) don’t vary much with $\theta$ (and hence $m$ in a Fourier expansion), another important property of the $f_{m+\tilde{m},n}$ Fourier coefficients is that they will be (at least approximately) translationally invariant:

$$f_{m+\tilde{m},n}(x) = e^{i\lambda} f_{m+\tilde{m},-1,n}(x - \Delta) = \cdots = e^{i\tilde{m} \lambda} f_*(x - \tilde{m} \Delta) \quad (110)$$

in which $\lambda \sim \pi/nq' \sim 1/n << 1$ is a small phase factor which will later be neglected. It represents a radial envelope [23]. Using this result and $\Delta \equiv 1/nq'$ in the representation of the nonaxisymmetric $f_{na}$ in (104) yields

$$f_{na} \approx \sum_{n \neq 0} e^{-in\alpha_n} \sum_{\tilde{m}=-\infty}^{\infty} f_*(\tilde{m} - n x q') e^{i\tilde{m} \theta + \lambda} \quad (111)$$

This is transformed using the Poisson sum formula [25]

$$\sum_{\tilde{m}=-\infty}^{\infty} F(\tilde{m}) = \sum_{p=-\infty}^{\infty} \hat{F}(2\pi p), \quad (112)$$

in which $\hat{F}(p)$ is the Fourier transform of $F(\tilde{m})$. The Fourier transform of the Fourier coefficient in (111) is

$$\hat{F}(p) \equiv \int_{-\infty}^{\infty} d\tilde{m} e^{i\tilde{m} \theta} \left[ f_*(\tilde{m} - n x q') e^{i\tilde{m} (\theta + \lambda)} \right]$$

$$= \int_{-\infty}^{\infty} dz e^{i(z + n x q') (\theta + \lambda + p)} f_*(z)$$

$$= e^{i n x q' (\theta + \lambda + p)} f_*(\theta + \lambda + p). \quad (113)$$

Using this result together with the Poisson sum formula (112) in (111) yields the representation

$$f_{na} \approx \sum_{n \neq 0} e^{-in\alpha_n} \sum_{p=-\infty}^{\infty} f_*(\theta + \lambda + 2\pi p) e^{i n x q' (\theta + \lambda + 2\pi p)}. \quad (114)$$

Note that this is a periodic function of both the poloidal angle $\theta$ and the helical angle $\alpha$. Since near a rational surface where $q \approx q_* + x q'$ the Clebsch coordinate $\beta \equiv \theta - \zeta$ is approximately $-\alpha + x q' \theta$, this form represents a locally periodic Clebsch magnetic field representation. Also, $\alpha \to \theta + \lambda + 2\pi p$ represents extending the poloidal angle $\theta$ into a field line variable along $\mathbf{B}$. [In the vicinity of a rational surface at a minimum in $q$ one has $q' = 0$; then, the Fourier modes in (114) are replaced by “modelets” [26, 27] and the modified ballooning transform theory proceeds qualitatively in the same manner as indicated here for the usual case where $q' \neq 0$.]

To put this “ballooning representation” in a more intuitive form it is useful to recognize that (for $n > > 1$) the distance $\ell$ along a helical magnetic field line is

$$\ell \approx (\theta + 2\pi p/\lambda) R_{q_*}, \quad \text{distance along } \mathbf{B}, \quad (115)$$

in which $-\pi \leq \theta \leq \pi$. Thus, converting the discrete sum over $p$ into a continuous integral over $\ell$, one obtains

$$f_{na} \approx \int_{-\ell_*}^{\ell_*} \frac{d\ell}{2\pi R_{q_*}} \hat{f}_*(\ell) e^{ik_\ell(x) \ell}. \quad (116)$$

Here, for the sheared ($q' \neq 0$) magnetic field,

$$k_\parallel(x) \equiv \frac{n x q'}{R_{q_*}}, \quad \text{effective parallel wavenumber.} \quad (117)$$

The limits on the $\ell$ integration in (116) are given by the half length of a helical field line $[\ell_* = \pi R_{q_*} n$ from (16)] because the radially diffusing finite $n$ helical field lines have a finite length; thus, the limits take account of the fact that the distribution function $\hat{f}_*(\ell)$ should be periodic over the length $2\ell_* = 2\pi R_{q_*} n$. These limits also imply that the sum over $p$ in (114) only extends from $-n/2$ to $n/2$ — to represent summing over $n$ poloidal transits of a helical field line in the vicinity of a $q_* = m/n$ rational flux surface. (Alternatively, one can think of them as summing over $m$ toroidal transits of a helical field line near the $q_* = m/n$ rational surface.) Since $\hat{f}_*(\ell)$ is usually nearly constant (see next subsection) for $|\ell| \leq \ell*$,
both (114) and (116) will yield factors of approximately \( \ell_s / \pi R q_s = n >> 1 \), which will produce the multiplier \( M \) [see (134) below] in the paleoclassical electron heat diffusivity — physically because contributions of \( n \) poloidal passes of the helical field line must be summed to obtain the net response for one poloidal period of the plasma. Note that the present analysis is only valid asymptotically since an \( n >> 1 \) approximation has been used throughout the derivation.

In the “ballooning representation” the parallel distance \( \ell \) is proportional to the Fourier transform variable \( k_x \) for the \( x \) (radial) variation of \( f_s(x) \). Also, one has

\[
k_{\parallel}(x) \ell = k_x(\ell) x, \tag{118}
\]

where \( k_x(\ell) = n q (\ell / R q_s) = n q (\theta + \lambda + 2 \pi p) \), which is \( k_{\theta} s \theta \) in usual ballooning mode theory [22–24] with \( k_{\theta} = n q / p a \) and \( s = p q' / q \).

Satisfaction of the criterion \( k^2 \delta_{\parallel}^2 < 1 \) (or \( |x|^2 > \delta_{\parallel}^2 \)) for diffusing helical magnetic field lines — see discussion between (75) and (76) — requires \( |\ell| < \ell_{\delta} \) where

\[
\ell_{\delta} = \frac{R q_s}{(n \delta_{\parallel} q')}, \quad \text{diffusing field line length.} \tag{119}
\]

This length will be equal to or longer than the helical field line length \( \ell_s = \pi R q_s n \) for \( n < n_{\max} \) in which

\[
n_{\max} = \frac{1}{(n \delta_{\parallel} q')^{1/2}}, \quad \text{maximum } n \text{ for } \ell_{\delta} > \ell_s. \tag{120}
\]

As after (22), if \( q' \) is negative it is to be replaced by its absolute value \( |q'| \). The length of helical field lines that are diffusing radially over their entire parallel length is

\[
\ell_{\max} = \pi R q_s n_{\max}, \quad \text{maximum diffusing length.} \tag{121}
\]

C. Nonaxisymmetric (Helically Resonant) Paleoclassical Transport

Solutions of the nonaxisymmetric MDKE (105) will be sought using an ordering scheme in which the transit frequency \( \omega_t \sim v_{\parallel} (B \cdot \nabla \theta) / B \sim v_{\parallel} c / R q \) is larger than all other frequencies. Thus, to lowest order the kinetic equation (105) is (near the \( q_s = m / n \) rational surface, assuming \( q - q_s \approx 0 \) \( \partial \ln f_{na} / \partial \alpha \approx |n x q'| = \left| \frac{x}{\Delta} \right| < 1 \))

\[
v_{\parallel} \frac{B \cdot \nabla \theta}{B} \left( \frac{\partial f_{s0}}{\partial \theta} \right)_{\psi, \alpha} = 0 \quad \Rightarrow \quad f_{s0}(\psi, \alpha, \varepsilon, \mu, t). \tag{122}
\]

That is, \( f_{s0} \) is independent of the poloidal angle \( \theta \) but depends on the helical angle \( \alpha \) and a radial coordinate \( \psi \) here. The next order kinetic equation includes parallel streaming along the \( \psi_s \) surfaces and collisions:

\[
v_{\parallel} \frac{B \cdot \nabla \theta}{B} \left( \frac{\partial f_{s1}}{\partial \theta} + (q - q_s) \frac{\partial f_{s0}}{\partial \alpha} \right) = C \{ f_{s0} \}. \tag{123}
\]

Bounce-averaging annihilates the \( \partial f_{s1} / \partial \theta \) term to yield

\[
\omega_t (q - q_s) \left( \frac{\partial f_{s0}}{\partial \alpha} \right)_{\psi_s} = \langle C \{ f_{s0} \} \rangle_{\theta}. \tag{124}
\]

Here, \( \langle \cdot \rangle_{\theta} \) indicates a bounce average,

\[
\langle A(\theta) \rangle_{\theta} = \frac{1}{\int B \sqrt{\theta} d\theta} \frac{A(\theta) / v_{\perp}}{B / \sqrt{\theta} d\theta / v_{\perp}} = \langle B A(\theta) / v_{\perp} \rangle / \langle B / v_{\perp} \rangle, \tag{125}
\]

and a parallel transit frequency has been defined:

\[
\omega_t (\varepsilon, \mu) = \frac{v_{\parallel} \sqrt{\mu}}{B / \sqrt{\theta} d\theta (B / v_{\perp})} = \frac{\langle B \rangle}{\langle B / v_{\perp} \rangle} R q, \tag{126}
\]

in which the average magnetic field strength \( \langle B \rangle \) along a field line is used as a normalizing factor and the average major radius \( R \) was defined in (17).

Since \( \partial \psi_s / \partial \rho = (q - q_s) \psi' \) and the operator on the left of (124) is proportional to the Poisson bracket term \( \{ \psi_s, f \}_{\rho, \alpha} \), the solution (for \( q - q_s \neq 0 \)) of the bounce-averaged equation (124) is a Maxwellian constant along the (slowly diffusing) \( \psi_s(\rho, t) \) helical flux surfaces:

\[
f_{s0} = f_M (\psi_s, \varepsilon, t) = n_e (\psi_s, t) \left( \frac{m_e}{2 \pi T_e (\psi_s, t)} \right)^{3/2} e^{-\varepsilon / T_e (\psi_s, t)}. \tag{127}
\]

This lowest order nonaxisymmetric equilibrium distribution is a Maxwellian that is a function of the helical flux function \( \psi_s \); hence, the electron temperature and density are constant on the \( \psi_s \) magnetic flux surfaces (i.e., along helical field lines). (Physically, the electron temperature is equilibrated along helical field lines by parallel electron heat conduction.) The parametric dependence of the electron density \( n_e \) and temperature \( T_e \) on the time \( t \) allows for temporal evolution of these quantities on the transport time scale, which is second order in the small gyroradius, em skin depth expansion.

It is important to understand the limits of applicability of the Maxwellian distribution in (127). This equilibrium solution is applicable on time scales longer than the electron collision time \( \tau > 1 / \nu_e \); hence it will be applicable on the transport time scale. The parallel length along a helical field line over which it applies can be inferred by considering the form of (124) for the transformed distribution \( f_{s0}(\ell) \). Substituting in the representation of \( f_{na} \) from (116) and using the fact that for this representation \( in x q' f_s(x) \rightarrow R q_s s f_{s0}(\ell) / \partial \ell \), (124) becomes

\[
\frac{\partial f_{s0}(\ell)}{\partial \ell} = \langle C \{ f_{s0} \} \rangle \tag{128}
\]

in which \( \tau_{\parallel} \equiv \omega_t R q_s = \langle B \rangle / \langle B / v_{\parallel} \rangle \). When the electron collision length \( \lambda_e \) is longer than the periodicity half length \( \ell_s = \pi R q_s n \) of the helical field line, an analysis similar to that used [5] for showing the lowest order axisymmetric distribution is a Maxwellian which is constant on the poloidal flux surfaces can be used to show that the appropriate solution in the vicinity of a rational surface is (127). When \( \lambda_e < \ell_s \), finite parallel electron heat conduction [28–30] limits the electron temperature equilibration to the region \( |\ell| \lesssim \lambda_e \). A more precise description that would determine the \( \ell \) dependence of \( f_{s0}(\ell) \)
in this regime, which requires a kinetic treatment similar to that developed in Refs. [29, 30] for calculating the parallel electron heat conduction in low collisionality toroidal plasmas, is beyond the scope of the present paper. Thus, the axial length scale over which the \( f_{\text{el}} \) distribution in (127) is applicable will be written as
\[
|\ell| \leq \ell_{\text{f,el}} \equiv \min \{ \ell_*, \lambda_c \}, \quad \text{for } f_{\text{el}} \text{ applicability.} \tag{129}
\]

The helically resonant radial electron heat transport induced by the radial diffusion of poloidal flux can now be determined. The contribution in the vicinity of a single \( q_s \equiv m/n \) rational surface will be determined first. It is obtained by taking the helical average of the energy moment of the \( D(f_{\text{el}}) \) term in (105), and using the \( f_{\text{el}} \) representation in (116) and the Maxwellian lowest order distribution in (127) subject to the limits given in (129):
\[
-\langle \nabla \cdot Q_{e*}^\text{PC} \rangle \equiv \left\langle \int_{-\pi}^{\pi} \frac{dx}{2\pi} e^{i\alpha x} \int d^3\mathbf{v} \frac{m v^2}{2} D(f_{\text{el}}) \right\rangle \\
\approx \int_{-L}^{L} d\ell \frac{e^{ik_1(x)\ell}}{2\pi Rq_s} \frac{3}{2} n_e T_e. \tag{130}
\]

The Fokker-Planck coefficients in (83) are only applicable for \( x^2 > R_c^2 \). Thus, as indicated in (121), the maximum half length of helical field lines that are diffusing radially over their entire length is \( \ell_{\text{max}} \). Hence, the limits of the \( \ell \) integration in (130) are given by \( \pm L \), in which \( L \) is the minimum of the length over which \( f_{\text{el}} \) is Maxwellian \((\ell_{\text{f,el}} \approx \ell_{\text{max}}}) and \( \ell_{\text{max}} \):
\[
L \equiv \min \{ \ell_{\text{max}}, \lambda_c, \ell_{n^0} \}, \quad \text{equilibration length.} \tag{131}
\]

Note that \( L \) is the minimum of three important lengths: the maximum length \( \ell_{\text{max}} \) of helical field lines diffusing radially over their entire length, the electron collision length \( \lambda_c \), and \( \ell_{n^0} \), which is the (short) half length of helical field lines near a low order surface where \( q^0 = m^0/n^0 \):
\[
\ell_{n^0} = \pi Rq_s n^0, \quad \text{low order rational line length.} \tag{132}
\]

For low order rational surfaces one would like to consider, for example, \( m/n = 2/1, 3/2, 3/1 \); however, rigorously speaking, the present analysis uses an asymptotic approximation valid only for \( n \gg 1 \). Thus, while qualitatively \( \ell \) is limited to a much shorter length in the vicinity of low order surfaces, quantitatively the relevant length is only of the order of \( \ell_{n^0} \) in (131).

Since very near a rational surface where \( |k_1(x)L| \ll 1 \), one can set \( e^{ik_1(x)\ell} \approx 1 \) for \( \delta_{q} \ll \ell \ll \pi Rq_s/nq \leq 1/\left(\pi n^2 q^2\right) = \Delta/\pi n \), perform the \( \ell \) integration in (130), and obtain for the total (advective plus diffusive) paleoclassical electron heat flux near the \( q_s \) flux surface
\[
\langle Q_{e*}^\text{PC} \cdot \nabla V \rangle = \kappa' M (a_g + \bar{\psi}) \frac{3}{2} n_e T_e - \frac{\partial}{\partial \rho} \left( V' M D_n \frac{3}{2} n_e T_e \right). \tag{133}
\]

Here, the multiplier \( M \) is
\[
M = \frac{L}{\pi Rq_s} \rightarrow \frac{L}{\pi Rq} \rho, \quad \text{helical multiplier.} \tag{134}
\]

Physically, recalling the discussion in the paragraph after (117), the geometric factor \( M \sim 10 \gg 1 \) enters because the distribution of electrons is Maxwellian over \( L \), which is a long distance (compared to the poloidal periodicity length \( \pi Rq_s \)), and as helical field lines advect and diffuse radially they carry this parallel-extended Maxwellian and equilibrated \( T_e \) with them. In a stationary poloidal field equilibrium \((\text{i.e., } \bar{\psi} = \psi = 0)\) and the rest frame of the toroidal flux surfaces \((\text{i.e., } u_q)\) the advective part of the paleoclassical flux \( Q_{e*}^\text{PC} \) vanishes.

Since the solutions \( f_{\text{el}}(\ell) \) are reasonably constant over the length range \( |\ell| \leq L \), (130) can be integrated leaving in the \( e^{ik_1(x)\ell} \) factor to estimate the radial response factor near a given rational surface that should be multiplied by the result given in (133):
\[
H_{\text{m/n}}(x) = \frac{\sin \left( k_1(x)L \right)}{k_1(x)L} = \frac{\sin \left( \frac{x}{x_L} \right)}{x/x_L}, \tag{135}
\]

in which the characteristic radial scale length for this autocorrelation-type function is
\[
x_L \equiv \frac{Rq_s}{L} n q^2 \geq \frac{1}{\pi} \frac{1}{n_{\text{max}} n q^2} = \frac{\delta x}{\pi}. \tag{136}
\]

The inequality follows from the definition of \( L \) in (131) as the minimum of three characteristic lengths. Note that for \( L = \pi Rq_s n_{\text{max}} \) the function \( H_{\text{m/n}}(x) \) vanishes at \( \left| x \right| = \delta x \), which is the radial distance from an \( m/n \) rational surface to the closest rational surface with \( n = n_{\text{max}} \), which was defined in (23).

In the usual paleoclassical “collisionless” case where \( \lambda_c > \ell_{\text{max}} \), when the \( f_{\text{el}} \) responses from adjacent rational surfaces are summed \((\sum m_n H_{\text{m/n}}(x)) \), they produce an overall heat flux that is nearly constant (to within about 10%) as \( x \) (or \( \rho \)) traverses many medium order \((n \ll n_{\text{max}} \sim 10)\) rational surfaces. (In summing over all possible rational surfaces the sum over \( n > n_{\text{max}} \) rational surfaces, on which the helical field lines are not diffusing over their entire length, yields a \( \sum_{n_{\text{max}} < n} n_{\text{max}}^2 / n_{\text{max}}^2 \approx n_{\text{max}} \) correction to the \( M \) given above, which could about double the net result in the collisionless case. Also, for collisional cases where \( \lambda_c < \ell_{\text{max}} \), the responses overlap radially somewhat and could produce a slightly larger \( M \) — an effect which however is likely compensated for by a reduction in \( f_{\text{el}} \) at large axial distances \( \ell \sim \lambda_c \) due to finite parallel electron heat conduction effects \([29, 30]\).)

Near a low order rational surface where \( n = n^0 = 1 \) or 2, the radial distance between the \( n^0 \) and \( n_{\text{max}} \) rational surfaces is larger \((\delta x^0 \approx 1/n^0 n_{\text{max}} q^2) \), and because \( L \) is much shorter for low order rational field lines, the \( M \) in (134) is much smaller over this larger radial range. This effect is particularly significant when \( q \) is at a minimum near a low order rational surface since then the corresponding rational surface separation is the larger \( \delta x \rightarrow \delta x_{\text{min}} \) in (24) with \( n \rightarrow n^0 = 1 \) or 2.

In view of all these considerations of radial profile effects, the net (after summing over all possible rational surfaces) helical paleoclassical electron heat flux varies.
little with radius from (133) with $q_s \rightarrow q(\rho)$ in (134), except in the vicinity of low order rational surfaces where it also varies little with radius, but is smaller because $L$ and hence $M$ are smaller there. However, in view of the approximations made and approximate limits used to obtain the net helical paleoclassical results, the factor $M$ should be considered a scaling result; more detailed studies are required to determine appropriate numerical, order unity “headache factors” that multiply $M$.

D. Near Magnetic Separatrix

In a divertor configuration as one moves radially outward and approaches the magnetic separatrix, many things happen: $q$ becomes very large ($\rightarrow \infty$ on the separatrix); the poloidal periodicity length $\pi Rq$ becomes very long and in particular longer than the collision length $\lambda_c$; the variation in the “straight-field-line” poloidal coordinate $\theta$ becomes highly concentrated near the X-point; and magnetic field lines become predominantly toroidal, especially near the X-point. While the preceding analyses, which implicitly assumed $\lambda_c > \pi Rq$, are no longer applicable, a related analysis can be performed in the near-separatrix region when $\pi R < \lambda_c < \pi Rq$.

Useful coordinates near a magnetic separatrix are $\rho$, $\hat{\theta} \equiv \zeta/q_s - \theta$ and $\zeta$. For large $q_s$, $\hat{\theta} \simeq \theta$. The magnetic field here can be written as $B = \nabla \hat{\theta} \times \nabla \psi + \nabla \zeta \times \nabla \psi \hat{\theta}$, with $\nabla \psi \equiv (1/q - 1/q_s) \nabla \psi_s$. In contrast to (106), now $\mathbf{B} \cdot \nabla f = (\mathbf{B} \cdot \nabla \zeta) \partial f/\partial \zeta - (1/q - 1/q_s) \partial f/\partial \hat{\theta}$. Performing an analysis similar to that for the axisymmetric (poloidal) case above (Section IV.B), but now based on toroidal periodicity, one obtains to lowest order $f_{\rho 0} \simeq f_M(\psi_0, \zeta, \tau)$ for $\partial f_{\rho 0} / \partial \hat{\theta} \ll 1$ and a near-separatrix (subscript $s$) paleoclassical heat flux

$$\langle Q^{pc}_{es} \cdot \nabla V \rangle = V'(\bar{u}_{\rho} + \bar{u}_{\psi}) \frac{3}{2} n_e T_e - \frac{3}{2} \frac{\partial}{\partial \hat{\theta}} \left( V' \eta^{sp}_{\eta} \frac{3}{2} n_e T_e \right).$$

The parallel resistivity here is the Spitzer resistivity $\eta^{sp}_{\eta}$ because this result is only applicable in the large $q$, near-separatrix region where $\pi R < \lambda_c < \pi Rq$, which is the high-collisionality Pfirsch-Schüller regime [5, 6] where electron viscosity effects are negligible ($\nu_{ee} / \nu_e < 1$). This result effectively extends the axisymmetric results in (101), (102) to the more collisional paleoclassical regime $\pi R < \lambda_c < \pi Rq$ near (but inside) a magnetic separatrix.

VII. ELECTRON ENERGY TRANSPORT

Adding together the axisymmetric (neoclassical plus paleoaclassical) and nonaxisymmetric (helically resonant) paleoclassical transport effects from (101) and (133), one obtains an overall electron energy balance equation in the rest frame of the toroidal flux surfaces (in $\mathbf{u}_g$ rest frame):

$$\frac{3}{2} \frac{\partial}{\partial \bar{V}} \left( n_e T_e V^{5/3} \right) + \frac{\partial}{\partial V} \left( \mathbf{e}^{pc} \cdot \nabla V \right)$$

$$\frac{\partial}{\partial \bar{V}} \left( \mathbf{e}^{pc} \cdot \nabla V \right) + \mathbf{Q}_{ec} + \mathbf{Q}_{es} = \mathbf{Q}_{g}.$$  (138)

The total paleoclassical electron heat flux ($\mathbf{Q}^{pc}_{ec}$) in the rest frame of the toroidal flux surfaces is the sum of (101) and (133) with the $\bar{u}_g$ terms removed:

$$\langle \mathbf{Q}^{pc}_{es} \cdot \nabla V \rangle = V'(M+1) \bar{u}_{\psi} \frac{3}{2} n_e T_e$$

$$- \frac{\partial}{\partial \rho} \left( V'(M+1) \bar{D}_0 \frac{3}{2} n_e T_e \right).$$

Note that for a stationary poloidal magnetic field ($\psi = 0$) there is no advective contribution to the paleoclassical electron heat flux $\mathbf{Q}^{pc}_{es}$ since then $\bar{u}_\psi = 0$.

The diffusive part of the electron heat flux in (139) indicates a paleoclassical electron heat diffusivity of

$$\chi^{pc}_e \equiv \frac{3}{2} (M+1) D_0 \approx \frac{3}{2} M \frac{\eta^{pc}_{\eta}}{\mu_0}, \quad \text{paleoclassical } \chi_e.$$

(140)

Comparing the paleoclassical electron heat diffusivity $\chi^{pc}_e$ with the poloidal magnetic flux diffusion in (65), one sees that $T_e$ diffuses a factor of order $M$ faster than $\psi$ does — because of the equilibrium of $T_e$ over the long length $L$ of the helical field lines, compared to the poloidal periodicity length $\pi Rq$. Note also that $\chi^{pc}_e$ is applicable only for weakly transient situations where $\psi$ satisfies (87).

There are two collisionality regimes of paleoclassical electron heat diffusion. They will be illustrated for a plasma whose poloidal magnetic field is in equilibrium ($\psi = 0$). For most toroidal plasmas the collision length $\lambda_c$ is longer than $\ell_{max}$; then, $M = n_{max} \gg 1$ yields

$$\chi^{pc}_e \approx \chi^{pc}_{\eta} \left( \frac{1}{\pi \delta_c (q')^1/2} \right)^{1/2} \frac{\eta^{pc}_{\eta}}{\mu_0}, \quad \ell_{max} \leq \ell_{max} = \pi Rq n_{max}. \quad (141)$$

As an example [1] of the magnitude of the “collisionless” $\chi^{pc}_e$, for a typical ohmically-heated Tokamak Fusion Test Reactor (TFTR) plasma [31] with $T_e \approx 1.2$ keV, $n_e \approx 3 \times 10^{19} \text{ m}^{-3}$, $Z_{eff} \approx 2$, $R_0 \approx 2.55 \text{ m}, q \approx 1.6$, and $\nu_{ee} / \nu_e \approx 0.4$ at $r/a \approx 0.4/0.8 = 0.5$, one obtains $\eta_0 / \mu_0 = 0.067 \text{ m}^2/\text{s}, \eta^{pc}_{\eta} / \mu_0 = 2.2$ (neglecting $\nu_{ee}$ effects, which would make the results a factor of 0.6 smaller), $\delta_c \approx 10^{-3} \text{ m}, n_{max} \approx 11$, and $\lambda_c \approx 0.3 \text{ m} > \pi Rq n_{max} \approx 140 \text{ m},$ so that $L \approx \pi Rq n_{max}, M = n_{max} \approx 11$, and the estimated $\chi^{pc}_e$ is 2.5 m$^2$/s $\sim \chi^{pc}_{\eta}$. The radial variation of this electron heat diffusion is due predominantly to the $T_e$ dependence of the resistivity, since in the “collisionless” regime $\chi^{pc}_e \propto n_e^{1/4} / (q' T_e)^3/2 \propto T_e (r)^{-3/2}$ — but $\nu_{ee}$ effects can be significant for $0.1 \lesssim \nu_{ee} \lesssim 3$. 
In high density plasmas where $L$ is determined by the electron collision length $\lambda_e$, $M = \lambda_e / \pi R_q >> 1$ yields
\[ \chi_e^{pc} \approx 3 \eta_e^{pc} \frac{v_{Te} e^2}{2 \eta_0 \pi R_q \omega_p^2}, \] collisional pc regime,
\[ \pi R_q < \lambda_e < \pi R_q n_{max}. \] (142)

The $\pi R_q < \lambda_e$ limit reflects the implicit assumption in the preceding helical paleoclassical kinetic analysis that the parallel equilibration length is longer than the poloidal periodicity length. In typical high density toroidal plasmas $Z_{eff} \approx 1$ and $\nu_{ce} >> 1$; for such plasmas $(3/2)(\eta_e^{pc}/\eta_0) \approx (3/2)(0.51)$. Thus, the collisional $\chi_e^{pc}$ implies an overall electron energy confinement time $\tau_{ee} \approx a^2 / 4 \chi_e^{pc} \approx 0.27 (n_e/10^{20} \text{m}^{-3}) a^2 R_0 (T_e/500 \text{eV})^{-1/2} \text{s}$, which approximately reproduces (in both magnitude and scaling for the highest performance pellet-fueled Alcator C plasmas [32] that had a $\approx 0.165 \text{m}$ and $R_0 = 0.64 \text{m}$) the “neo-Alcator scaling” deduced empirically primarily from ohmically-heated tokamak plasma data in the 1970s and early 1980s [33]; $\tau_{ee}^{A} \approx 0.07 n_e a R_0 T_{9}^{A}$. The collisional regime $\chi_e^{pc}$ in (142) is similar to a $\chi_e$ formula originally proposed by Okawa [34]. Okawa’s model and subsequent studies by Kadomtsev and Pogutse [35] were based on a combination of an empirical formula for $\chi_e$ that could reproduce the neo-Alcator confinement scaling law [32, 33] and a heuristic model of the collisionless effects of turbulence-induced “magnetic flutter” [19], possibly due to drift waves, within the em skin depth layer ($|x| < \delta_e$). Thus, these other models were based on very different physics than the resistivity-induced magnetic line diffusion outside the em skin depth layer ($|x| > \delta_e$) and collisional regime ($\lambda_e$-limited) paleoclassical model presented here.

In the near-separatrix region where $q$ and $q'$ become very large, one can have $\lambda_e \ll \pi R_q$. In this region, combining the results in (137) and (139), the paleoclassical electron heat diffusivity becomes
\[ \chi_e^{pc} \approx 3 \eta_e^{pc} \frac{v_{Te} e^2}{2 \eta_0 \pi R_q \omega_p^2} \left(1 + \frac{\eta_e^{pc} \lambda_e}{\eta_0^{pc} \pi R_q} \right), \]

near separatrix,
\[ \pi R < \lambda_e < \pi R_q n_{max} \{1, n_{max}\}. \] (143)

For $\lambda_e / \pi R_q > (\eta_0^{pc} / \eta_e^{pc}) \approx 1$ this yields the collisional (Alcator-scaling) $\chi_e^{pc}$ in (142). In the opposite limit, which applies where $q$ gets very large near the separatrix, one obtains a smaller $\chi_e^{pc} \approx (3/2)(\eta_0^{pc} / \mu_0) \approx [100/(T_e(eV))^{3/2}] \text{m}^2/\text{s}$ for $Z_{eff} \approx 1$. There are some experimental indications in Doublet-III D-shaped (DIII-D) plasmas [36, 37] that within about 2 cm of the magnetic separatrix the electron temperature gradient is significantly larger, which implies $\chi_e^{pc}$ is reduced there; the maximum $T_e$ gradient apparently occurs at about the $\rho^* \approx 0.95-0.97$ predicted by the paleoclassical model: $q(\rho^*) \approx (\lambda_e / \pi R_q) (\eta_0^{pc} / \eta_0) \approx 5-10$.

The paleoclassical model derived in this paper applies to all types of axisymmetric toroidal plasmas — tokamaks, spherical tokamaks (STs), spheronaks, and reversed field pinches (RFPs) — in regions where $c^2, B_0^2 / B^2 << 1$. For one meter major radius STs with $T_e \approx 1 \text{keV}$ and $n_e \approx 3 \times 10^{19} \text{m}^{-3}$, the prediction at $r/a \approx 0.5$ is $\chi_e^{pc} \approx 5-10 \text{m}^2/\text{s}$, which, in reasonable agreement with experimental results [38, 39], is large because for STs $q' << 1$ is small and $\eta_e^{pc} / \eta_0 \approx 3$ is large in the plasma confinement region ($r/a \approx 0.5$). For quiescent RFP plasmas such as those in the Madison Symmetric Torus (MST) Pulsed Poloidal Current Drive (PPCD) experiments [40, 41], at $r/a \approx 0.3-0.5$ one obtains $\chi_e^{pc} \approx 5-10 \text{m}^2/\text{s}$ (large because $q < 0.2$ and $|q'| \approx 0.2$ are small), which is close to the effective $\chi_e$’s inferred from global $\chi_e^{exp} \approx a^2 / 4 \tau_E \approx 7.5 \text{m}^2/\text{s}$ [40] and local $\chi_e^{exp} \approx 10-30 \text{m}^2/\text{s}$ [41] measurements. In quasi-symmetric stellarator plasmas there would be no paleoclassical transport if there is no flux-surface-average parallel current ($J \cdot B$) in the plasma; however, net flux-surface-average parallel currents in a stellarator would induce a $\chi_e^{pc}$.

As indicated by (131), (134), and (140), the predicted $\chi_e^{pc}$ is much smaller in the vicinity of low order rational surfaces with $q^o = m/n^o$: $\chi_e^{pc} \approx (3/2)(n^o + 1) \eta_e^{pc} / \mu_0$. Over what radial width does this “transport barrier” exist? From the discussion in the next to last paragraph in Section VI.C, the width of the transport barrier is
\[ 2 \delta x^e \approx \frac{2}{n^o} \left( \frac{\pi \delta_e}{|q'|} \right)^{1/2}, \text{ barrier width}. \] (144)

Or, if $q$ is very near a minimum at the rational surface,\[ 2 \delta x^e_{min} \approx 2 \left( \frac{2}{n^o} \right)^{2/3} \left( \frac{\pi \delta_e}{|q'|} \right)^{1/3}, \text{ width at } q_{min}. \] (145)

These estimated transport barrier widths can be used to interpret some key electron heat transport experimental results. First, as experiments in Rijnhuizen Tokamak Project (RTP) plasmas [42] slowly moved highly localized electron cyclotron heating (ECH) radially outward, a “stair-step” reduction in the central $T_e$ was observed as the ECH passed low order rational surfaces. From these results it was inferred [42] that internal transport barriers (ITBs) existed in which the electron heat diffusivity was reduced by up to a factor of 10 over relative (to a) barrier widths of order 0.04 (0.1 for $q = 1/1$). For the RTP parameters, (144) predicts $2 \delta x^e \approx 0.06-0.12$ (0.17 for $q = 1/1$), a bit wider but in reasonable agreement with the experimental results. Next, jumps in $T_e$ (over radial zones of width $\approx 0.2$) have been observed in evolving DIII-D L-mode plasmas [43] as an off-axis minimum in $q(p, t)$ passes through low order rational surfaces. For the relevant DIII-D parameters, $2 \delta x^e_{min}$ gives a similar estimate ($\approx 0.3$) for the barrier width. Finally, in pioneering Japan Torus 60 Upgrade (JT-60U) experiments [44] an electron ITB created a large $\nabla T_e$; the experimentally observed barrier width was $2 \delta x \approx 0.11$, close to the paleoclassical prediction from (145) of 0.14, and $\chi_e^{pc}$ is in approximate agreement (within experimental error bars) with $\chi_e^{exp}$ outside, within, and inside the electron ITB.
The paleoclassical electron heat flux in (139) is not in a normal (diffusive) Fourier heat flux law form (i.e., \( \mathbf{q}_e = -\kappa_e \mathbf{V}_e T_e \equiv -n_e \chi_e \mathbf{V}_e T_e \)). Rather, it can be written (for \( \psi = 0 \)) in terms of (normal, Fourier) diffusive and (abnormal) “heat pinch” fluxes:

\[
\langle \mathbf{q}^e_{pe} \cdot \mathbf{V} \rangle = -V' n_e \chi_e \mathbf{V}_e T_e \frac{\partial T_e}{\partial \rho} - \langle \mathbf{q}^e_p \cdot \mathbf{V} \rangle, \tag{146}
\]

\[
\langle \mathbf{q}^e_p \cdot \mathbf{V} \rangle \equiv T_e \frac{\partial}{\partial \rho} \left( V' n_e \chi^e_{pe} \right), \quad \text{heat pinch.} \tag{147}
\]

This electron heat pinch heat flux is usually positive (inward) and increases with \( \rho \), in qualitative agreement with experimental inferences from Joint European Torus (JET) \[45\] and recent DIII-D \[46\] experiments. Alternatively, in qualitative agreement with some tokamak experimental data \[47\], \( \langle \mathbf{Q}^e_{pe} \cdot \mathbf{V} \rangle \) can be written in the form of a heat flux proportional to the degree to which \( -\mathbf{V} \ln T_e \equiv 1/L_{Te} \) exceeds a critical magnitude:

\[
- \frac{1}{L_{Te}} \nabla_e \left. \right| \approx \frac{\partial}{\partial \rho} \ln(V' n_e \chi^e_{pe}), \quad \text{critical gradient of } \ln T_e. \tag{148}
\]

In tokamak plasmas with ohmic and modest auxiliary electron heating, \( \nabla \ln T_e \) in the “confinement region” \( (0.3 \lesssim \rho \lesssim 0.8) \) is nearly constant \[48\], and is usually close to its critical value. The paleoclassical model would be in agreement with such experimental results if the critical gradient of \( \ln T_e \) in (148) is approximately constant over the confinement region.

For paleoclassical electron heat transport the effective or “power balance” \( \chi_e (\chi^e_{pe}) \), which is defined as the net electron heat flux divided by \( -n_e \mathbf{V}_e T_e \) (based on an assumed Fourier heat flux law), is

\[
\chi^e_{pe} = \frac{\langle \mathbf{Q}^e_{pe} \cdot \mathbf{V} \rangle}{-V' n_e \mathbf{V}_e T_e \frac{\partial}{\partial \rho}} = \chi^e_e \left( 1 - \frac{\partial \ln(V' n_e \chi^e_{pe})/\partial \rho}{-\partial \ln T_e/\partial \rho} \right). \tag{149}
\]

In the usual situation where \( T_e \) decreases as the minor radius variable \( \rho \) increases, but \( V' n_e \chi^e_{pe} \) increases with \( \rho \), one obtains \( \chi^e_{pe} < \chi^e_e \).

In tokamak plasmas, electron temperature profiles \( T_e(\rho) \) often seem to be related to the \( q(\rho) \) profile and are hard to change — properties often called “profile resilience” \[45\], which was initially called “profile consistency” \[31, 49\]. For ohmically heated plasmas (where the transformer-induced flux changes induce both toroidal current and ohmic heating), since \( \chi^e_{pe} \) is a multiple \( M \) of the magnetic field diffusivity \( D_M \) and it depends explicitly on \( q(\rho) \), the paleoclassical model may be able to explain the experimentally observed \( T_e \) profile resilience.

There has been a long-standing mystery \[50–52\] as to why transiently measured electron heat diffusivities usually exceed those determined from equilibrium “power balance” measurements. A straightforward perturbation \( (T_e \rightarrow T_{eo} + \delta T_e) \) of the electron energy balance equation (138) would yield a “heat pulse” \( (\text{superscript } hp) \) electron heat diffusivity \( \chi^e_{pe} \) that is negative for the collisionless paleoclassical regime \( (141) \) where \( \chi^e_{pe} \propto T_e^{-3/2} \). However, transient electron heat transport induced by radially localized perturbations can be considered within the present paleoclassical model only for cases where \( \psi \) satisfies conditions (87). Many transiently measured \( \chi_e \)'s use sawtooth-generated heat pulses or localized electron heating to induce radially localized transients in \( T_e \). These effects change the local \( T_e \) and hence the resistivity \( \eta^e_{pe} \); from the definition of \( \psi \) in (80), they are very likely to violate conditions (87), particularly the first one. Further, since for \( \psi \neq 0 \) these transient experiments change the toroidal inductive electric field in the plasma, they also change the \( \mathbf{E} \cdot \mathbf{J} \) joule heating embodied in the \( Q_e \) collisional heating term \[5, 6\] in the electron energy balance equation (138). A T-10 experiment \[53\] that surely violated conditions (87) observed a large inward “ballistic jump in the total heat flux just after on-axis ECRH switching on.” A full transient transport paleoclassical analysis that self-consistently includes all possible effects due to rational flux surface motion, local helical flux distortions, inductive electric field perturbation effects, and changes in the joule heating is beyond the scope of the present paper. However, the profound question that results from this discussion is: can one ever measure the local electron heat diffusivity via electron heating or localized transient responses? The answer seems to be: only if \( T_e \) can be changed in experiments that cause small, relatively homogeneous poloidal magnetic flux \( (\psi) \) perturbations that satisfy conditions (87).

**VIII. SUMMARY AND DISCUSSION**

As emphasized in the first part of Section VI, the key hypothesis of the paleoclassical model is that the electron guiding centers advect and diffuse radially with the same Fokker-Planck coefficients as the poloidal magnetic flux does, namely (83). The main results obtained from this fundamental hypothesis are in (139)–(149). They are supplemented by definitions of the: multiplier \( M \) in (134), equilibration length \( L \) in (131), magnetic field diffusivity \( D_M \) in (66), and neoclassical parallel resistivity \( \eta^e_{pe} \) in (46)–(53). For transients they are applicable only for a slowly changing (in space and time) poloidal flux \( \psi \) that satisfies (87). Since the results were obtained using a large asymptotic analysis and the characteristic lengths in \( L \) were only approximately determined, the formula for \( M \), and hence all \( M \)-dependent results herein, should be interpreted as scaling results; more detailed studies beyond the scope of this paper could introduce numerical coefficients of order unity in \( L, M \), and \( \chi^e_{pe} \).

The results in (138) through (142) reduce to those obtained previously \[1\] with a slab model in the limit of a large aspect ratio tokamak when the poloidal magnetic field is stationary. The previous paper \[1\] elucidated how the resultant, simplest paleoclassical model provides interpretations for many aspects of the experimentally ob-
served electron heat diffusivity: magnitude, multiple of magnetic field diffusivity $D_n$, radial profile, collisionality regime, Alcator scaling law in high density tokamak plasmas, and reduced $\chi_e$’s near low order rational surfaces and just inside a magnetic separatrix.

The main contributions of this paper have been to: 1) Develop the paleoclassical transport model for a full axisymmetric toroidal magnetic field geometry (assuming $e^2B_p^2/B_t^2 << 1$); 2) Illustrate the evolution equations for the poloidal magnetic flux $\psi$, the toroidal flux $\psi_*$ and the local helical flux $\psi_\ast$, and their interrelations; 3) Develop an appropriate parallel Ohm’s law for the paleoclassical model that includes neoclassical and inertial effects; 4) Clarify the local diffusion properties of the poloidal flux $\psi$ and, near rational flux surfaces, the helical flux $\psi_\ast$; 5) Determine the limits on $\psi$ in (87) for application of the paleoclassical model to transient electron heat transport studies; 6) Explain how electron gyromotion radially across poloidal flux surfaces combined with radial diffusion of poloidal flux (field lines) leads to radial diffusion of the usual $p_e \propto \psi$ constant of the motion and hence of the electron guiding center; 7) Develop a low collisionality kinetic analysis of paleoclassical processes based on adding spatial Fokker-Planck coefficients to the electron drift-kinetic equation to take account of the electron guiding center diffusion induced by the radial diffusion of the poloidal magnetic flux; 8) Explore the parallel extent of the Maxwellianization of the electron distribution and $T_e$ equilibration along poloidal and helical field lines; 9) Determine the axisymmetric and helically resonant radial electron heat flux induced by the diffusing poloidal flux; 10) Clarify the ballooning-type representation of the nonaxisymmetric (helically resonant) distribution function which yields the multiplier $M$ that reflects the long distance along field lines over which $T_e$ is equilibrated; 11) Estimate the radial variation of $\chi_{e}^{pc}$ and width of “internal transport barriers” where $\chi_{e}^{pc}$ is significantly reduced in the vicinity of low order rational surfaces; and 12) Illustrate how heat pinch or minimum electron temperature gradient forms emerge naturally from the paleoclassical electron heat flux.

Because paleoclassical electron heat transport is based on the primitive Coulomb collision processes of parallel electron heat conduction and electrical resistivity leading to magnetic field diffusion, it should be considered an “irreducible, ubiquitous” plasma transport process, just as classical and neoclassical transport [5, 6] are. Plasma turbulence induced by microinstabilities could induce additional transport. Since $D_n \propto \eta \propto 1/T_e^{3/2}$, “collisionless” paleoclassical electron heat diffusion decreases with increasing electron temperature; for $n_{\max} \lesssim 10$ and $\eta_0^{PC}/\eta_0 \lesssim 2$, $\chi_e^{PC}$ becomes less than 1 m$^2$/s for $T_e \gtrsim 2$ keV. Thus, microturbulence-induced transport due to trapped-electron or electron-temperature-gradient instabilities, which typically result in a gyroBohm scaling of $\chi_{\text{turb}}^{\ast} \propto (q/a)(T_e/eB) \sim T_e^{-3/2}$, could become the dominant electron heat transport mechanisms for electron temperatures above a few keV. Such microturbulence would apparently not affect paleoclassical processes since it usually does not significantly affect the parallel Ohm’s law [54, 55] and the parallel correlation length for magnetic microturbulence usually exceeds the relevant paleoclassical parallel length $L$.

The preceding section discussed the promising interpretations of “anomalous” electron heat transport provided by the the toroidal paleoclassical model presented in this paper for tokamaks, spherical tokamaks (STs) and quiescent reversed field pinch (RFP) plasmas. And for tokamak plasmas, the paleoclassical theory provides models for radial widths of transport barriers near low order rational $q$ minima, in near-separatrix regions, and heat pinch or minimum temperature gradient forms of the electron heat flux. These reasonably successful comparisons with experimental results plus those cited in the original paleoclassical paper [1] encourage further development of the paleoclassical model, and more detailed and definitive comparisons with experimental data — to see to what degree the observed “anomalous” radial electron heat transport in low collisionality toroidal plasmas can be definitively explained by the paleoclassical model.

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