Paleoclassical electron heat transport

J.D. Callen
University of Wisconsin, Madison, WI 53706-1609, USA
E-mail: callen@engr.wisc.edu, http://homepages.cae.wisc.edu/~callen

Abstract. It is hypothesized that radial electron heat transport in low collisionality, magnetically-confined toroidal plasmas results from paleoclassical Coulomb collision processes (parallel electron heat conduction and magnetic field diffusion). In such plasmas the electron temperature equilibrates along magnetic field lines a long length $L$, which is the minimum of the electron collision length and a maximum effective half length of helical field lines. Thus, the diffusing field lines induce a radial electron heat diffusivity $M \approx L/(\pi R_0 \eta) \approx 10 >> 1$ times the magnetic field diffusivity $\eta/\mu_0 \approx \nu_e (e/\omega_p)^2$. The paleoclassical electron heat flux model provides interpretations for many features of “anomalous” electron heat transport: magnitude and radial profile of electron heat diffusivity (in tokamaks, STs, and RFPs), Alcator scaling in high density plasmas, transport barriers around low order rational surfaces and near a separatrix, and a natural heat pinch (or minimum temperature gradient) heat flux form.

PACS numbers: 52.25.Fi, 52.35.Vd, 52.55.Dy, 52.55.Fa

1. Introduction: paleoclassical physical mechanism

The fastest (earliest) and most primitive (“paleo”), dominant Coulomb-collision-induced transport processes in magnetically-confined plasmas occur on the electron collision time scale $1/\nu_e$ and will be called paleoclassical [1,2]: parallel electron heat conduction and magnetic field diffusion. On this time scale, the electron distribution is Maxwellianized and electron heat conduction equilibrates the electron temperature $T_e$ over long distances parallel to the magnetic field $B$ — up to the electron collision length $\lambda_e \equiv v_{Te}/\nu_e$ in which $v_{Te} \equiv (2T_e/m_e)^{1/2}$. Magnetic field diffusion is induced by the plasma electrical resistivity $\eta$. It causes magnetic flux (bundles of field lines) to diffuse perpendicular to $B$ with a diffusion coefficient $D_\eta \approx \eta_0/\mu_0 \equiv \nu_e (e/\omega_p)^2 \sim (\Delta x)^2/\Delta t$, which implies a diffusive radial step $\Delta x \approx \delta_e \equiv c/\omega_p$ [the electromagnetic (em) skin depth] in a collision time $\Delta t \approx 1/\nu_e$.

Electron gyromotion about magnetic field lines causes the electron guiding center to be identified with the small amount of magnetic flux associated with field lines penetrating the gyroorbit. However, since those field lines diffuse radially due to $D_\eta$, it is hypothesized that the electron guiding center position becomes a radially diffusing “stochastic variable.” To account for this effect a spatial Fokker-Planck operator [3] is
added to the usual drift-kinetic equation — see (33), (34). If $\lambda_e$ is longer than the length of a helical field line on a $q_\ast \equiv m/n$ rational surface or the effective parallel length of diffusing rational field lines [for $n \leq n_{\text{max}} \sim 10$ — see (40)], the parallel equilibration length $L$ is reduced to these lengths — see (44). The effect of the $T_e$ equilibration over a long length $L$ along radially diffusing helical rational field lines that are longer than the poloidal periodicity half length ($\sim \pi R_0 q$) is that the effective electron heat diffusivity is a multiple $M \sim L/(\pi R_0 q) \sim 10$ of the magnetic field diffusivity $D_\parallel$ — see (39), (43), (46), and (50).

The paleoclassical model for radial electron heat transport was first introduced using a sheared slab magnetic field model [1]. Thereafter, it was developed in some detail for an axisymmetric magnetic magnetic field geometry [2]. The key results of those papers are elucidated and summarized in the first four sections of this paper: 2. Magnetic field geometry, 3. Magnetic flux, field line diffusion, 4. Paleoclassical kinetics, analysis, and 5. Paleoclassical radial electron heat transport. In Section 5 a number of new encouraging comparisons of the paleoclassical model for radial electron heat transport with experimental data are discussed. Section 6 discusses the range of applicability and limitations of the paleoclassical model, and suggests a number of local transport and fundamental tests of it. The final section provides a very brief summary of this paper.

2. Magnetic field geometry

The paleoclassical model will be developed here using a full axisymmetric magnetic field model for arbitrary aspect ratio ($A \equiv R_0/r \equiv 1/\epsilon$ where $R_0$, $r$ are the major, minor radii of the torus) to facilitate application of the theory to most types of axisymmetric toroidal plasmas — large aspect ratio tokamaks ($A >> 1$) and regions of spherical tokamaks (STs, $A \gtrsim 1$), spheromaks, and reversed field pinches (RFPs) where $\epsilon^2 B_p^2/B_t^2 << 1$. Approximate results for large aspect ratio tokamaks are indicated at the end of many equations after an approximate equality ($\sim$).

Paleoclassical transport is concerned with diffusion of magnetic flux (bundles of magnetic field lines). Since for axisymmetric toroidal plasmas with $\epsilon^2 B_p^2/B_t^2 << 1$ the toroidal magnetic flux $\psi_t$ is less mobile than the poloidal magnetic flux $\psi$ [4-6], diffusion of the poloidal flux surfaces (and field lines) will be determined relative to $\psi_t$ and hence a dimensionless, cylindrical-type radial variable $\rho$: $\rho \equiv [\psi_t/\psi_t(a)]^{1/2} \simeq r/a$, $\psi_t(\rho, t) \equiv (1/2\pi) \int dS(\zeta) \cdot B_t \simeq r^2 B_0/2$. The appropriate magnetic field model [4-6] has toroidal ($t$) and poloidal ($p$) components: $\mathbf{B} = \mathbf{B}_t + \mathbf{B}_p = I \mathbf{\nabla} \zeta + \mathbf{\nabla} \zeta \times \mathbf{\nabla} \psi = \mathbf{\nabla} \psi \times \mathbf{\nabla} (q \theta - \zeta)$. As usual, $I = I(\rho, t) \equiv R B_t \simeq B_0 R_0$. Also, $\zeta$ is the toroidal angle and $\psi(\rho, t) \equiv (1/2\pi) \int dS(\theta) \cdot \mathbf{B}_p$, $\partial \psi/\partial \rho \simeq a R_0 B_p$. Further, $\theta$ is the straight-field-line (in the $\psi = \text{constant plane}$) poloidal angle and $q$ is the winding number or pitch (“safety factor” for kink stability) of magnetic field lines on a flux surface: $q(\rho, t) \equiv (\partial \psi_t/\partial \rho)/(\partial \psi/\partial \rho) = \# \text{toroidal transits}/\# \text{poloidal transits} \simeq r B_t/R_0 B_p$. For an axisymmetric magnetic field $q(\rho, t) = q(\psi, t)$ and $\mathbf{B} \cdot \mathbf{\nabla} \theta = I/q R^2 \simeq B_t/R_0 q = B_p/r$.

The Jacobian for transforming from the original Eulerian coordinates to the
Paleoclassical electron heat transport

curvilinear set \( u^i \equiv (\rho, \theta, \zeta) \) is \( \sqrt{g} \equiv 1/\mathbf{\nabla} \rho \cdot \mathbf{\nabla} \theta \times \mathbf{\nabla} \zeta = (\partial \psi / \partial \rho) / (\mathbf{B} \cdot \mathbf{\nabla} \theta) \sim r a R_0 \). The radical differential of the volume is \( V' \equiv \partial V(\rho, t) / \partial \rho = 2 \pi \int_{-\pi}^{\pi} \sqrt{g} \, d\theta \sim a(2 \pi r) (2 \pi R_0) \).
The average of an axisymmetric \((\partial f / \partial \zeta = 0)\) scalar function \( f(x, t) \) over a flux surface is \( f(x, t) = (2 \pi / V') \int_{-\pi}^{\pi} \sqrt{g} \, d\theta \, f(x, t) \). The flux-surface-average is an annihilator for the parallel gradient operator: \( \langle \mathbf{B} \cdot \mathbf{\nabla} f \rangle = 0 \), for any function \( f(x, t) \) that is periodic in both \( \theta \) and \( \zeta \). For a similarly periodic vector field \( \mathbf{A}(x, t) \), the flux-surface-average of its divergence, defined by \( \mathbf{\nabla} \cdot \mathbf{A} \equiv \sum_i (1/\sqrt{g}) (\partial / \partial u^i)(\sqrt{g} \mathbf{A} \cdot \mathbf{\nabla} u^i) \), becomes \( \langle \mathbf{\nabla} \cdot \mathbf{A} \rangle = \partial (\mathbf{A} \cdot \mathbf{\nabla} V) / \partial V \).

Flux surfaces are rational or irrational depending on whether \( q \) is the ratio of integers \( (m, n) \):

\[
q(\rho, t) \begin{cases} 
  = m/n, & \text{rational surface,} \\
  \neq m/n, & \text{irrational surface.} 
\end{cases}
\]

The irrational surfaces form a dense set while the rational surfaces are a set of measure zero and radially isolated from each other. Rational surfaces are of interest here because their (helical) magnetic field lines close on themselves after \( m \) toroidal (or \( n \) poloidal) transits.

The differential length \( d\ell \) along magnetic field lines obtained from the poloidal \( (\mathbf{\nabla} \theta) \) projection of the field line equation \( dx / d\ell = \mathbf{B} / B \) is \( d\ell = (B / \mathbf{B} \cdot \mathbf{\nabla} \theta) \, d\theta \sim R_0 q \, d\theta \). The half length \( \ell_* \) of a closed helical field line on a \( q_* \equiv q(\rho_*) \equiv m/n \) rational surface is [2]:

\[
\ell_* \equiv \frac{1}{2} \int_{-\pi}^{\pi} \frac{B \, d\theta}{\mathbf{B} \cdot \mathbf{\nabla} \theta} = \pi \tilde{R} q_* n, \quad q_* \text{ line length,} 
\]

\[
\tilde{R} \equiv \frac{1}{2 \pi q_* \partial \psi / \partial \rho} \int_{-\pi}^{\pi} \sqrt{g} \, d\theta \, B = \frac{\langle B \rangle \, V'}{4 \pi^2 q_* \partial \psi / \partial \rho} \sim R_0. \quad (3)
\]

While helical field lines on medium order rational surfaces with \( n \sim 10 >> 1 \) are long \((\sim \pi \tilde{R} q_*)\), those with low \( n \) \((\equiv n^0 = 1, 2)\) are short \((\sim \pi \tilde{R} q_*)\).

Radial distances between medium order rational surfaces can be estimated using a Taylor series expansion of \( q(\rho, t) \) about a rational surface at \( \rho = \rho_* \):

\[
q(\rho, t) \simeq q_* + x \, q' + O(x^2), \quad q' \equiv |\partial q / \partial \rho|_{\rho_*}, 
\]

\[
q_* \equiv q(\rho_*, t) = m/n, 
\]

\[
x \equiv \rho - \rho_. 
\]

The distance between rational surfaces with \( m \pm 1 \) but the same \( n \) is obtained from \( 1/n = q - q_* \sim x q' \):

\[
\Delta \simeq 1/n q', \quad \text{same } n \text{ rational surface spacing.} 
\]

Defining \( q(\rho_{\text{max}}) = m_{\text{max}} / n_{\text{max}} \) and expanding \( q(\rho) = (m_{\text{max}} n + 1) / n_{\text{max}} n \) about \( \rho = \rho_{\text{max}} \) yields the distance between a \( q_* \equiv m/n \) rational surface and the nearest \( n \leq n_{\text{max}} \) rational surface:

\[
\delta x(n) \equiv \rho_* - \rho_{\text{max}} \simeq \frac{1}{n_{\text{max}} n q'}, 
\]
which is the minimum spacing for \( q' \neq 0 \), \( n \leq n_{\text{max}} \). At a minimum in \( q \) where \( q' \) vanishes, one obtains

\[
\delta x_{\text{min}}(n) \equiv \rho_{*} - \rho_{\text{max}} \simeq \left( \frac{2}{n_{\text{max}} n q''} \right)^{1/2},
\]

in which \( q'' \equiv \frac{\partial^2 q}{\partial \rho^2} \big|_{\rho_{*}} \). For \( n_{\text{max}} \gtrsim 10 \), \( q' \sim 1 \), and \( q'' \sim 1 \), all of these distances are small fractions of the minor radius: \( \Delta \sim 1/n_{\text{max}} \ll 1 \) for \( n \sim n_{\text{max}} \), and \( \delta x(n) \lesssim 1/n_{\text{max}} \ll 1 \), \( \delta x_{\text{min}}(n) \lesssim 1/\sqrt{n_{\text{max}}} < 1 \).

3. Magnetic flux, field line diffusion

Evolution equations for the toroidal flux \( \psi_t \), and and the poloidal flux \( \psi \) obtained from Faraday’s law \( \frac{\partial \mathbf{B}}{\partial t} = -\mathbf{\nabla} \times \mathbf{E} \) are [2,5,6]:

\[
\frac{d\psi_t}{dt} = \left. \frac{\partial \psi_t}{\partial t} \right|_{\mathbf{x}} + \langle \mathbf{u}_g \cdot \mathbf{\nabla} \psi_t \rangle = 0,
\]

(10)

\[
\frac{d\psi}{dt} = \left. \frac{\partial \psi}{\partial t} \right|_{\mathbf{x}} + \langle \mathbf{u}_g \cdot \mathbf{\nabla} \psi \rangle = \frac{\langle \mathbf{E} \cdot \mathbf{B} \rangle}{\langle \mathbf{B} \cdot \mathbf{\nabla} \zeta \rangle} - \frac{\partial \Psi}{\partial t},
\]

(11)

\[
\langle \mathbf{u}_g \cdot \mathbf{\nabla} \psi_t \rangle \equiv q \frac{\langle \mathbf{E} \cdot \mathbf{B} \rangle}{\langle \mathbf{B} \cdot \mathbf{\nabla} \zeta \rangle}, \quad \text{“grid velocity.”}
\]

(12)

The toroidal flux \( \psi_t \) is advected radially by the grid velocity \( \mathbf{u}_g \) induced by the poloidal electric field, but conserved in a Lagrangian frame. In (11) \( \partial \Psi / \partial t \equiv V_{\text{loop}}^c(t)/2\pi \) is the (positive) constant of a spatial integration. It represents the toroidal loop voltage induced by the rate of change of the magnetic flux in the central solenoid of a tokamak.

The poloidal flux \( \psi \) and hence poloidal magnetic field lines move relative to \( \psi_t \) [compare (10) and (11)] because of departures from ideal MHD (i.e., a nonzero parallel electric field \( \langle \mathbf{E} \cdot \mathbf{B} \rangle \)) or a temporally changing magnetic flux in the central solenoid (i.e., \( \partial \Psi / \partial t \neq 0 \)).

A parallel Ohm’s law for \( \langle \mathbf{E} \cdot \mathbf{B} \rangle \) is obtained [2] from the flux-surface-average of the parallel \( (\mathbf{B} \cdot \mathbf{\nabla}) \) component of the electron momentum equation including inertial and viscosity effects:

\[
\frac{\langle \mathbf{E} \cdot \mathbf{B} \rangle}{\langle \mathbf{B} \cdot \mathbf{\nabla} \zeta \rangle} = \left( \frac{n_{\text{nc}}}{\mu_0} + \delta_e^2 \frac{d}{dt} \right) \frac{\langle \mu_0 \mathbf{J} \cdot \mathbf{B} \rangle}{\langle \mathbf{B} \cdot \mathbf{\nabla} \zeta \rangle} + \frac{\mu_e}{\nu_e} \eta_0 \frac{1}{\langle R^{-2} \rangle} \frac{dP}{d\psi}.
\]

(13)

Here, the terms on the right indicate: magnetic flux diffusion [see (21) below] induced by the neoclassical parallel resistivity, electron inertia, and the neoclassical bootstrap current. The (neoclassical) parallel electrical resistivity (neglecting poloidal electron heat flow effects) is

\[
\frac{\eta_{\text{nc}}}{\eta_0} \simeq \frac{\eta_{\text{Sp}}}{\eta_0} + \frac{\mu_e}{\nu_e}, \quad \text{neoclassical resistivity,}
\]

(14)

in which \( \eta_{\text{Sp}} \) is the classical parallel resistivity:

\[
\frac{\eta_{\text{Sp}}}{\eta_0} \simeq \frac{\sqrt{2} + Z}{\sqrt{2} + 13Z/4}, \quad \text{Spitzer resistivity.}
\]

(15)
Paleoclassical electron heat transport

The reference (⊥) resistivity \( \eta_0 \) and electron viscous drag frequency \( \mu_e \) adapted from [5,6] are

\[
\frac{\eta}{\mu} \equiv \frac{m_e \nu_e}{n_e e^2 \mu_0} \simeq 1.4 \times 10^3 Z \left( \frac{\ln \Lambda}{17} \right) \frac{m^2}{s}, \tag{16}
\]

\[
\frac{\mu_e}{\nu_e} \simeq \frac{Z + \sqrt{2} - \ln (1 + \sqrt{2})}{Z (1 + \nu_e^{1/2} + \nu_e)} \frac{\nu_e = 0}{f_c} \simeq 1.5 \frac{f_t}{f_c}. \tag{17}
\]

Here, \( Z \ (\rightarrow Z_{\text{eff}} \equiv \sum_i n_i Z_i^2 / n_e \) for multiple ion species) is the (effective) ion charge, and \( f_c \simeq 1 - 1.46 e^{1/2} + \mathcal{O}(e^{3/2}) \) is the circulating particle fraction [6], for which the fraction of trapped particles is \( f_t \equiv 1 - f_c \). The electron collisionality parameter is defined by

\[
\nu_{ce} \equiv \frac{\nu_e}{e^{3/2} (v_T e / R_q)} = \frac{R_q}{e^{3/2} \lambda_e}, \tag{18}
\]

in which the electron collision length \( \lambda_e \) is given by

\[
\lambda_e \equiv \frac{v_T e}{\nu_e} \simeq 1.2 \times 10^{16} \left[ \frac{T_e (eV)}{n_e Z} \right]^2 \left( \frac{17}{\ln \Lambda} \right) \text{ m.} \tag{19}
\]

The \( \eta^\perp \) in (14), (17) ranges [2] from being equal to (for \( \mu_e / \nu_e << 1 \)) to twice (for \( \mu_e / \nu_e >> 1 \)) the most precise neoclassical resistivity results [5,6].

From Ampere’s law, \( \mu_0 J = \nabla \times B = (\partial I / \partial \psi) \nabla \psi \times \nabla \zeta + \nabla \zeta \Delta^* \psi \), in which the usual magnetic differential operator is \( \Delta^* \psi \equiv (1 / |\nabla \zeta|^2) \nabla \cdot |\nabla \zeta|^2 \nabla \psi \). Dotting this \( \mu_0 J \) with \( B \), flux surface averaging, and using \( |\nabla \zeta|^2 = R^{-2} \) yields [2,6]

\[
\Delta^* \psi \equiv \frac{\langle \mu_0 J \cdot B \rangle}{\langle B \cdot \nabla \zeta \rangle} = \frac{I}{(R^{-2}) V^2 \partial r} \frac{\partial}{\partial \rho} \left( \frac{\left| \nabla \rho \right|^2}{R^2} \right) \frac{V^2 \partial \psi}{I \partial \rho} \simeq \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial \psi}{\partial r}. \tag{20}
\]

Substituting the Ohm’s-law-determined \( \langle E \cdot B \rangle \) in (13) using \( \langle J \cdot B \rangle \) from (20) into the poloidal flux evolution equation (11), one obtains a diffusion-type (at least for \( \delta^2 \Delta^* << 1 \)) equation for \( \psi \):

\[
\frac{d}{dt} (1 - \delta^2 \Delta^*) \psi = D_\eta \Delta^* \psi - S_\psi, \tag{21}
\]

\[
D_\eta \equiv \frac{\eta^\perp}{\mu_0}, \quad \text{magnetic field diffusivity.} \tag{22}
\]

Sources of poloidal flux in \( S_\psi \equiv \partial \Psi / \partial t - (\mu_e / \nu_e) \eta_0 (1 / (R^{-2})) \partial P / \partial \psi \) arise from the “current-drive” effects due to a changing flux in the central solenoid \( (\partial \Psi / \partial t \equiv V_{\text{loop}}^c / 2\pi) \) and the bootstrap current. An extra, noninductive current source \( J_S \) would add a term \( \eta^\perp \langle J_S \cdot B \rangle / I (R^{-2}) \) to the source \( S_\psi \). The magnetic flux diffusion and paleoclassical transport processes discussed below will occur whenever \( D_\eta \Delta^* \psi \propto \eta^\perp \langle J \cdot B \rangle \neq 0 \). Thus, they will occur even if a noninductive current source causes the toroidal loop voltage to vanish since then the noninductive current source just replaces inductive source \( \partial \Psi / \partial t \).
In equilibrium in the Lagrangian frame, \( d/dt \to 0 \) and the equation for the stationary poloidal flux \( \psi_0 \) becomes \( 0 = D_\eta \Delta^+ \psi_0 - S_\psi \). Thus, in equilibrium the diffusion of \( \psi \) (poloidal flux, field lines) is balanced by the source \( S_\psi \) of poloidal flux; the Poynting flux represented by \( \partial \Psi/\partial t \) brings poloidal field lines into the plasma and the magnetic field diffusivity \( D_\eta \) diffuses them out of the plasma — even for a stationary poloidal magnetic field \( \mathbf{B}_p \), which will be henceforth assumed.

To determine the properties of the small bundle of poloidal magnetic flux \( \delta \psi(x,t) \) penetrating an electron gyroorbit, one substitutes an Ansatz of \( \delta \psi \equiv \delta \psi_0 + \delta \psi \) into (21) to obtain (for \( x^2 << 1 \)) \( (\partial/\partial t + \bar{u}_g \partial/\partial x)(1 - \delta^2 \partial^2/\partial x^2) \delta \psi \approx \bar{v}_e \delta^2 \partial^2 \delta \psi/\partial x^2 \), in which the following normalized variables have been defined:

\[
\bar{u}_g \equiv \langle \mathbf{u}_g \cdot \nabla \rho \rangle, \quad \bar{\delta}_e \equiv \frac{\delta_e}{\bar{a}}; \quad \bar{1}/\bar{a}^2 \equiv \frac{1}{\langle R^{-2} \rangle} \left( \frac{|\nabla \rho|^2}{R^2} \right) \approx \frac{1}{a^2}, \quad (23)
\]

\[
\bar{D}_\eta \equiv \frac{D_\eta}{\bar{a}^2}, \quad \bar{\nu}_e \equiv \frac{\nu_e}{\eta_0}. \quad (24)
\]

For \( x < \delta_e \) (or \( k_x^2 \delta^2_e > 1 \)), the \( \delta \psi \) solution of this equation is spatially constant [2]; hence, it produces no field lines or diffusion of them in this region (i.e., \( \delta \mathbf{B}_p \equiv \nabla \zeta \times \nabla \delta \psi = 0 \) there).

For radial scale lengths longer than the em skin depth \( \delta_e \) (for which \( k_x^2 \delta^2_e << 1 \) so that \( \delta^2 \Delta^+ << 1 \) can be neglected), the evolution equation for \( \delta \psi \) becomes a simple diffusion equation:

\[
\frac{d \delta \psi}{dt} \equiv \left( \frac{\partial}{\partial t} + \bar{u}_g \frac{\partial}{\partial x} \right) \delta \psi = \bar{D}_\eta \frac{\partial^2 \delta \psi}{\partial x^2}. \quad (26)
\]

Its (Green function) solution for a small bundle of flux initially located at \( x = x_0 \) (\( > \bar{\delta}_e \)), which is represented by \( \delta \psi(x,t = 0) = \delta \psi_0 \delta(x - x_0) \), is

\[
\delta \psi(x,t) = \delta \psi_0 \frac{e^{-(x-x_0-\bar{u}_g t)^2/4\bar{D}_\eta t}}{(4\pi \bar{D}_\eta t)^{1/2}}. \quad (27)
\]

Note that \( \delta \psi \) indicates a temporally evolving probability distribution for the radial location of the \( \delta \psi \) bundle of poloidal flux (field lines) that was initially at \( x = x_0 \). The mean displacement and radial spread (variance) of this bundle of magnetic flux are

\[
\int_{-\infty}^{\infty} dx \frac{(x-x_0) \delta \psi}{\int_{-\infty}^{\infty} dx \delta \psi} = \bar{u}_g t, \quad (28)
\]

\[
\int_{-\infty}^{\infty} dx \frac{(x-x_0)^2 \delta \psi}{\int_{-\infty}^{\infty} dx \delta \psi} = 2\bar{D}_\eta t = 2\bar{v}_e t \delta^2_e. \quad (29)
\]

As indicated, the average radial displacement and spread (variance) of this small amount of poloidal flux (bundle of field lines) grow linearly with time. Note that this magnetic flux (field line) advection and diffusion process occurs even when the poloidal magnetic field \( \mathbf{B}_p \) is in stationary equilibrium (i.e., \( d\psi/dt = 0 \)), which is being assumed here.
In the next section a Fokker-Planck model will be used to include effects of radial advection and diffusion of field lines in a kinetic analysis. Relevant Fokker-Planck coefficients deduced from (27) are (for $x^2 > \delta_x^2$)

$$\frac{\langle \Delta x \rangle}{\Delta t} = \bar{u}_g, \quad \frac{\langle (\Delta x)^2 \rangle}{2 \Delta t} = \bar{D}_\eta. \quad (30)$$

The Fokker-Planck coefficients can be written in a general vectorial form in terms of the covariant base vector in the “radial” direction $e_\rho \equiv \partial x/\partial \rho = \sqrt{g} \nabla \theta \times \nabla \zeta$, for which $e_\rho \cdot \nabla \rho = 1$:

$$\frac{\langle \Delta x \rangle}{\Delta t} = \frac{\langle \Delta x \rangle}{\Delta t} e_\rho, \quad \frac{\langle \Delta x \Delta x \rangle}{\Delta t} = \frac{\langle (\Delta x)^2 \rangle}{\Delta t} e_\rho e_\rho. \quad (31)$$

To consider diffusion of helical flux (field lines) in the vicinity of a rational surface at $\rho = \rho_*$ where $q_* \equiv q(\rho_*) = m/n$, one uses a local helical coordinate system $u^i \equiv (\rho, \theta, \alpha)$ with helical angle $[7] \alpha \equiv \zeta - q_* \theta = \zeta - (m/n) \theta$. Since $\nabla \theta \times q_* \nabla \theta = 0$, the Jacobian $\sqrt{g} \equiv (\nabla \rho \cdot \nabla \theta \times \nabla \alpha)^{-1}$ is the same as before. Thus, one writes $B$ in the local helical form $B = \nabla \alpha \times \nabla \psi + \nabla \psi_\star \times \nabla \theta \equiv B_h + B_\star$, in which the helical flux $\psi_\star$ is defined by

$$\partial \psi_\star/\partial \rho = (q - q_*) \partial \psi/\partial \rho. \quad (32)$$

Near a rational surface using $q(\rho) \simeq q_* + x q'$ one obtains $\psi_\star \simeq (x^2/2) q' \psi'$. Integrating the general form in (32) over $\rho$ near $\rho = \rho_*$, taking its total time derivative, and using $d\psi_\star/\Delta t = 0$ from (10), one obtains $[2]$ (again neglecting em skin depth effects for $x^2 > \delta_x^2$) and $t > 1/\bar{v}_e$ $d\psi_\star/\Delta t = -q_* d\psi/\Delta t$. Thus, the helical flux $\psi_\star$ diffuses like the poloidal flux $\psi$ does. Also, writing $\psi_\star \rightarrow \psi_\star + \delta \psi_\star$, one finds $[2]$ that $\delta \psi_\star$ obeys the same diffusion-type equation as $\delta \psi$ does, i.e., (26). Hence helical flux (field lines) in the vicinity of rational surfaces also advects and diffuses with the Fokker-Planck coefficients given in (31).

4. Paleoclassical kinetics, analysis

In drift-kinetic theory the magnetic flux surfaces and field lines are usually assumed to be fixed in space; however, as discussed in the preceding section, when $\eta \langle J \cdot B \rangle \neq 0$ they diffuse radially. The question then becomes: how do the electrons, and in particular their guiding centers, respond to the diffusing poloidal magnetic field lines? Because of the axisymmetry, the canonical momentum in the toroidal direction (with covariant base vector $e_\zeta \equiv R^2 \nabla \zeta$) is a constant of the “collisionless” electron motion:

$$p_\zeta \equiv e_\zeta \cdot (m_e v + q_e A) = m_e v_\zeta + q_e A_\zeta = \text{constant}. \quad \text{For the axisymmetric toroidal magnetic field being used here, } A_\zeta = e_\zeta \cdot (-\psi \nabla \zeta) = -\psi. \quad \text{Thus, averaging over the electron gyroperiod } (2\pi/\omega_{ce} \text{ with } \omega_{ce} \equiv q_e B/m_e), \text{ the constant of the electron guiding center motion can be written as } -p_\zeta/q_e = \psi - v_\parallel RB_t/\omega_{ce}. \quad \text{[The corresponding canonical helical momentum has the same form: } p_\alpha \equiv e_\alpha \cdot p = m_e v_\alpha - q_e \psi \Longrightarrow -p_\alpha/q_e = \psi - R\bar{v}_\alpha/\omega_{ce}.\]

Paleoclassical transport $[5,6]$ results from the stochastic diffusion of the $v_\parallel$ term in $p_\zeta$ due to the diffusion in velocity space induced by Coulomb collisions which is represented
in the drift-kinetic equation by the Fokker-Planck collision operator. The key hypothesis of paleoclassical theory is that, as a corollary, the spatial advection and diffusion of small magnetic flux bundles (such as those penetrating the electron gyroorbit) carry the electron guiding centers with them. Thus, they cause $p_\zeta$ (and $p_\alpha$) and hence the electron guiding center position to become a stochastic, diffusing variable. That is, the Fokker-Planck spatial advection and diffusion coefficients given in Eq. (31) represent not just the motion of small flux bundles but also the motion of the electron guiding centers, at least for $x^2 > \delta_e^2$.

The relevant electron kinetic equation is the gyro-averaged one, which is called the drift-kinetic equation [5]. Adding the Fokker-Planck-type effects [3] of the radial diffusion of the electron guiding centers induced by the magnetic flux (field line) diffusion, the magnetic-field-diffusion-Modified Drift-Kinetic Equation (MDKE) is

$$\frac{\partial f}{\partial t} + v_g \cdot \nabla f + \dot{\varepsilon} \frac{\partial f}{\partial \varepsilon} = \mathcal{C}\{f\} + \mathcal{D}\{f\}. \quad (33)$$

Here, $f = f(x, \varepsilon, \mu, t)$ is the guiding center distribution function, $v_g \equiv dx_g/dt = u_\parallel B/B + v_p$ is the guiding center velocity, $\mathcal{C}\{f\}$ is the Coulomb collision operator, and the other notation is standard.

Effects due to magnetic field line advection and diffusion of the electron guiding centers are indicated by the Fokker-Planck spatial diffusion operator ($\mathcal{D}$), which in general is [3] $\mathcal{D}\{f\} \equiv -\nabla \cdot [(\Delta x/\Delta t)f] + \nabla \cdot [(\Delta x\Delta x/2 \Delta t)f]$. Using $\nabla \cdot \mathbf{A} \equiv \sum_i(1/\sqrt{g})(\partial / \partial u) (\sqrt{g} \mathbf{A} \cdot \nabla u)$ and the Fokker-Planck coefficients in (31), when $f$ is solely a function of a magnetic flux coordinate (i.e., $\rho$, $\psi$, or $x$), the flux-surface-average of this operator becomes [neglecting $\langle \nabla \rho \cdot \partial \mathbf{e}_\rho / \partial \rho \rangle = \langle \nabla \rho \cdot \partial^2 \mathbf{x} / \partial \rho^2 \rangle \sim \mathcal{O}(\epsilon^2)]

$$\langle \mathcal{D}\{f(\rho)\} \rangle \simeq \frac{1}{V'} \frac{\partial}{\partial \rho} \left( -V' \dot{a}_g f + \frac{\partial}{\partial \rho} V' \mathcal{D}_\rho f \right). \quad (34)$$

Next, consider Fourier expansion of the distribution function in poloidal ($\theta$) and toroidal ($\zeta$) angles:

$$f(\psi, \theta, \zeta) = \sum_{m,n} f_{mn}(\psi) e^{im\theta-im\zeta} = \sum_m f_{m0}(\psi) e^{im\theta} + \sum_{m,n \neq 0} f_{mn}(\psi) e^{im\theta-im\zeta} \equiv f_a + f_{na}. \quad (35)$$

The $n = 0$ contributions represent the axisymmetric distribution function $f_a$ that yields the usual neoclassical transport [5,6]. The electron energy transport equation including both neoclassical and axisymmetric paleoclassical effects is obtained from the flux-surface-average of the kinetic energy moment of the axisymmetric part of (33), approximating $f$ in $\mathcal{D}\{f\}$ by a Maxwellian $f_M(\psi)$:

$$\frac{3}{2} \frac{\partial}{\partial t} \langle n_e T_e \rangle + \frac{\partial}{\partial V} \langle (\mathbf{q}_e^{nc} + \frac{5}{2} T_e \Gamma_e^{nc}) \cdot \nabla V \rangle + \frac{\partial}{\partial V} \langle \mathbf{Q}_e^{pc} \cdot \nabla V \rangle = Q_e. \quad (36)$$
Paleoclassical electron heat transport

Here, the electron entropy-producing processes are: the neoclassical conductive \( (q^\text{nc}_e) \) and convective \( [(5/2)T_o\Gamma^\text{nc}_e] \) heat fluxes, the paleoclassical heat flux \( (Q^\text{pc}_e) \) which is induced by \( D\{f_M\} \), and the heating \( (Q_e) \) due to collisional effects (joule heating, electron viscosity, and collisions with ions).

Near a \( q_\ast = m/n \) rational surface the nonaxisymmetric distribution function \( f_{na} \) can be put into a form that isolates its poloidal \( (\theta) \) and helical \( [\alpha \equiv \zeta - q_\ast \theta = \zeta - (m/n)\theta] \) angle dependences: \( f_{na}(\psi, \theta, \alpha) = \sum_{n \neq 0} e^{-ina} \sum_{m} f_{m+n,m,n}(\psi) e^{im\theta} \). Further, since near a rational surface the magnetic field can be represented by its helical and magnetic shear components as \( \mathbf{B} = \mathbf{B}_h + \mathbf{B}_s \), the parallel-streaming differential operator in (33) becomes [7]: \( \mathbf{B} \cdot \nabla f = (\mathbf{B} \cdot \nabla \theta) [\partial f/\partial \theta]_{\psi,\alpha} + (q - q_\ast) \partial f/\partial \alpha_{\psi,\theta} \). Thus, near the \( q_\ast \equiv m/n \) rational surface \( f \rightarrow f(\psi, \theta, \alpha, \varepsilon, \mu) \) and applying this parallel-streaming operator to \( f_{na} \) yields

\[
\mathbf{B} \cdot \nabla f_{na} = (\mathbf{B} \cdot \nabla \theta) \sum_{n \neq 0} e^{-ina} \sum_{m} e^{im\theta} [m - n(q - q_\ast)] f_{m+n,m,n}(\psi). \tag{37}
\]

Since the parallel-streaming term \( (v_{||}/B) \mathbf{B} \cdot \nabla f_{na} \) is dominant in (33), it causes the Fourier coefficients \( f_{m+n,m,n} \) to be small unless \( m - n(q - q_\ast) \) is small. Near the \( q_\ast \equiv m/n \) rational surface \( q \simeq q_\ast + xq' \) and this coefficient becomes \( m - n(q - q_\ast) \simeq m - nxq' \). It will be small and lead to the largest \( f_{m+n,m,n} \) for \( m = 0 \) and \( |nxq'| << 1 \). The resulting “helically resonant” Fourier coefficient (near \( q = q_\ast \)) will be \( f_\ast(x) \equiv f_{m,n}(\psi) \). Here, the argument has been changed from the poloidal \( (\psi) \) to the helical \( (\psi_\ast) \) flux, which is the appropriate flux (radial) label near the given rational surface. Using (7), the criterion \( |nxq'| << 1 \) is \( |x| << \Delta \). Hence, \( f_\ast(x) \) solutions will be highly peaked near the \( q_\ast \) rational surface.

Developing a useful (i.e., one-dimensional) representation for \( f_{na} \) near a \( q_\ast = m/n \) rational surface for \( n \gg 1 \) is analogous to the development of ballooning mode theory [8,9]. The basic issue is: how does one maintain periodicity of the solutions in the poloidal \( (\theta) \) and helical \( (\alpha) \) angles as one moves radially away (i.e., to \( x \neq 0 \) in a sheared magnetic field structure) from a rational surface composed of helically symmetric field lines. For “flute-like” behavior extending long distances \(|\ell| >> \pi R q_\ast \) along large \( n \) helical field lines, one assumes \( q \) is locally a linear function of \( x \) (i.e., \( q \simeq q_\ast + xq' \)) and employs the procedure Lee and Van Dam [9] used to develop a ballooning representation, to obtain [2]

\[
f_{na} \simeq \sum_{n \neq 0} e^{-ina} \sum_{p = -\infty}^{\infty} \hat{f}_\ast(\theta + 2\pi p) e^{inxq'(\theta + 2\pi p)}. \tag{38}
\]

Here, \( \hat{f}_\ast(\theta + 2\pi p) \) is [9] the Fourier transform of \( f_\ast(x) \). Note that this form of \( f_{na} \) is a periodic function of both the poloidal \( (\theta) \) and helical \( (\alpha) \) angles. In the limit of large \( n \) the discrete sum over \( p \) can be converted into a continuous integral over \( \ell \simeq (\theta + 2\pi p) R q_\ast \).
which represents extension of the poloidal angle $\theta$ into a field line variable along $B$:

$$f_{na}(x, \alpha) \simeq \sum_{n \neq 0} e^{-i \alpha x} \int_{-\ell_*}^{\ell_*} \frac{d\ell}{2\pi R q_*} \hat{f}_*(\ell) e^{i k_{||}(x) \ell}. \tag{39}$$

For the sheared ($q' \neq 0$) magnetic field $k_{||}(x) \equiv n x q' / R q_*$. The $\ell$ integration limits in (39) are the half length of a helical field line: $\ell_* = \pi R q_* n$ from (2). These limits also imply the sum over $p$ in (39) only extends from $-n/2$ to $n/2$ — to represent summing over $n$ poloidal transits of the field line. Since $\hat{f}_*(\ell)$ is usually nearly constant for $|\ell| \leq \ell_*$ (see discussion below (44) and [2]), (39) yields a factor $\sim \ell_* / \pi R q_* = n >> 1$, which produces the multiplier $M$ [see (46), (50)] in the paleoclassical electron heat diffusivity — physically because contributions of $n$ poloidal passes of the rational helical field line are summed to obtain the net response for one poloidal period of the plasma. In the “ballooning representation” the parallel distance $\ell$ is proportional to the Fourier transform variable $k_x(\ell)$ for the $x$ (radial) variation of $f_*(x)$. Also, note that $k_{||}(x) \ell = k_x(\ell) x$, where $k_x(\ell) \equiv n q'(\ell / R q_*), n q' (\theta + 2\pi p)$, which is the usual [8,9] $k_x = k_0 s \theta$ with $k_0 \equiv n q' / \rho a$ and $s \equiv \rho q' / q$.

Satisfying the criterion $k_x^2(\ell) \delta_x^2 < 1$ (or $|\ell|^2 > \delta_x^2$) for diffusing field lines requires $|\ell| < \ell_\delta \equiv R q_*/(n \delta_x q')$. Requiring $\ell_\delta$ to be longer than the helical field line length $\ell_* = \pi R q_* n$ in (2) yields a maximum $n$ and length of field lines that are diffusing over their entire length:

$$n_{\max} \equiv 1/(\pi \delta_x q')^{1/2}, \text{ maximum } n, \quad \ell_{\max} \equiv \pi R q_* n_{\max}, \text{ maximum diffusing length.} \tag{40} \tag{41}$$

Solutions of the nonaxisymmetric MDKE in (33) are sought [2] using an ordering scheme in which the transit frequency $\omega_t \sim v_{||}(B \cdot \nabla \theta) / B \sim v_T e / R q \theta$ is larger than all other frequencies. To lowest order $\partial f_{s0} / \partial \theta \big|_{\psi_*, \alpha} = 0$; hence $f_{s0}$ must be independent of the poloidal angle $\theta$. The next order kinetic equation includes parallel streaming along $\psi_*$ surfaces and collisions. Bounce-averaging it annihilates a $\partial f_{s1} / \partial \theta$ term to yield $\omega_t (q - q_s) \partial f_{s0} / \partial \alpha \big|_{\psi_*} = \langle C \{ f_{s0} \} \rangle_{\theta}$, whose solution [2] (for $q - q_s \neq 0, \lambda_e > \ell_*$) is a Maxwellian constant along $\psi_*$ surfaces (i.e., closed field lines with the pitch of rational field lines):

$$f_{s0} = n_e(\psi_*, t) \left( \frac{m_e}{2\pi T_e(\psi_*, t)} \right)^{3/2} e^{-\varepsilon / T_e(\psi_*, t)}. \tag{42}$$

When $\lambda_e < \ell_*$, finite parallel electron heat conduction [10,11] limits the electron temperature equilibration to the region $|\ell| \lesssim \lambda_e$. Thus, the $f_{s0}$ in (42) is applicable for $|\ell| \leq \ell_{f_{s0}} \equiv \min \{ \ell_*, \lambda_e \}$.

The paleoclassical radial electron heat transport induced by the diffusion of the nonaxisymmetric, helical magnetic flux (field lines) near $q_* = m/n$ is obtained by taking flux-surface-average of the helical average of the energy moment of the $D\{ f_{na} \}$ term in
Paleoclassical electron heat transport

(33) and using the \( f_{na} \) representation in (39) with \( f_{s0} \) from (42):

\[
- \langle \nabla \cdot \mathbf{Q}_{\text{pc}}^e \rangle \equiv \left\langle \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{ina} \int d^3v \frac{mv^2}{2} D\{f_{na}\} \right\rangle \simeq \left\langle D\{\int_{-L}^{L} \frac{dl}{2\pi R_{q*}} e^{ik_\parallel(x)\ell} \frac{3}{2} n_e T_e \} \right\rangle.
\]

(43)

The Fokker-Planck coefficients in (31) are only applicable for \( x^2 > \delta_e^2 \). Thus, the maximum half length of helical field lines is the \( \ell_{\text{max}} \) in (40). Hence, the limits of the \( \ell \) integration in (43) are given by \( \pm L \), in which \( L \) is the minimum of the lengths over which \( f_{0s} \) is Maxwellian (\( \ell_{M,s} \)) and field lines are diffusing (\( \ell_{\text{max}} \)):

\[ L \equiv \min \{ \ell_{\text{max}}, \lambda_e, \ell_{\text{eq}} \}, \text{ equilibration length.} \]  

(44)

Since very near a rational surface \( |k_\parallel(x)L| << 1 \), one can set \( e^{ik_\parallel \ell} \simeq 1 \) [for \( |x| << R_{q*}/nq' L \leq 1/(\pi n^2 q') = \Delta/\pi n \)], perform the \( \ell \) integration in (43), and obtain for the total (advective plus diffusive) paleoclassical electron heat flux near the \( q_s \) flux surface

\[
\langle \mathbf{Q}_{\text{pc}}^e \cdot \mathbf{\nabla} V \rangle = V' M \bar{u}_g \frac{3}{2} n_e T_e - \frac{\partial}{\partial \rho} \left( V' M \bar{D}_v \frac{3}{2} n_e T_e \right),
\]

(45)

\[ M \equiv \frac{L}{\pi R_{q*}}, \text{ helical multiplier.} \]  

(46)

Considering radial profile effects [2], the net helical paleoclassical electron heat flux (after summing over all possible rational surfaces) varies little with radius. Thus, \( q_s \) in (46) can be replaced by \( q(\rho) \).

In the vicinity of low order rational surfaces where \( q^\circ \equiv q(\rho^\circ) = m^\circ/n^\circ \) with \( n^\circ = 1, 2 \), the factor \( M \) also varies little with radius. However, it is then smaller in magnitude because \( L \) and hence \( M \) are smaller there. The factor \( M \) remains small as \( q \) increases away from the rational surface until a distance of order \( \delta_e \) from the closest \( n_{\text{max}} \) rational surface is reached. This distance can be estimated (neglecting \( \delta_e << 1 \) compared to \( \delta_e^{1/2} \)) from (8),

\[
\delta x^\circ \equiv \delta x(n^\circ) = \frac{1}{n^\circ} \left( \frac{\pi \delta_e}{|q'|} \right)^{1/2},
\]

(47)

or, around a minimum in \( q \) about \( q^\circ \) from (9),

\[
\delta x^\circ_{\text{min}} \equiv \delta x_{\text{min}}(n^\circ) = \left( \frac{2}{n^\circ} \right)^{2/3} \left( \frac{\pi \delta_e}{q''} \right)^{1/3}.
\]

(48)

5. Paleoclassical radial electron heat transport

Neoclassical [4-6] and other transport fluxes are usually specified in the rest frame of the toroidal flux surfaces. The paleoclassical heat flux will be specified in this rest frame
by removing the \( u_q \) contributions. For a stationary poloidal magnetic field \( \mathbf{B}_p \), the total paleoclassical electron heat flux \( Q_e^{pc} \) is the sum of axisymmetric (\( M \rightarrow 1 \) in (46) [2]) and the nonaxisymmetric (quasi-helically-symmetric) transport flux in (46):

\[
\langle Q_e^{pc} \cdot \nabla V \rangle = -\frac{\partial}{\partial \rho} \left( V'(M+1) \frac{\eta_{e}^{pc}}{\mu_0} \frac{3}{2} n_e T_e \right). \tag{49}
\]

The diffusive part of this total paleoclassical electron heat flux indicates a paleoclassical electron heat diffusivity of

\[
\chi_e^{pc} \equiv \frac{3}{2} (M+1) D_\eta \approx \frac{3}{2} M \frac{\eta_{e}^{pc}}{\mu_0} = \frac{3}{2} M \nu_e \delta_e^2. \tag{50}
\]

Comparing this \( \chi_e^{pc} \) with the magnetic flux diffusivity \( D_\eta \) in (21), one sees that \( T_e \) diffuses a factor of order \( M \) faster than \( \psi \) does — because \( T_e \) is equilibrated over the long length \( L \) of helical field lines, compared to the poloidal periodicity length \( \pi R_q \).

Thus, the paleoclassical model may be able to explain the experimentally observed \( T_e \) “profile resiliency” [12], which was originally called “profile consistency” [13,14] and has often been linked to the \( q \) profile. Also, the paleoclassical electron heat transport will not come to its steady-state value until the poloidal magnetic field comes into equilibrium (on the “skin diffusion” time scale \( \tau_{sk} \sim a^2/6D_\eta \)).

There are two collisionality regimes of paleoclassical electron heat diffusion. For most toroidal plasmas the electron collision length \( \lambda_e \) is longer than \( \ell_{max} \); then, \( M = n_{max} >> 1 \) yields

\[
\chi_e^{pc} \approx \frac{3}{2} \left( \frac{1}{\pi \delta_e |q'|} \right)^{1/2} \frac{\eta_{e}^{pc}}{\mu_0}, \quad \lambda_e > \ell_{max} \equiv \pi R_q n_{max}, \tag{51}
\]

which will referred to as the collisionless paleoclassical regime. As an example of the magnitude of this “collisionless” electron heat diffusivity, for a typical ohmically-heated TFTR plasma [14] with \( T_e \approx 1.2 \text{ keV}, n_e \approx 3 \times 10^{19} \text{ m}^{-3}, Z_{eff} \approx 2, R_0 \approx 2.55 \text{ m}, q \approx 1.6, \text{ and } a/q' \approx 0.4 \text{ m at } r/a \approx 0.4/0.8 = 0.5, \) one obtains \( \eta_0/\mu_0 \approx 0.067 \text{ m}^2/\text{s}, \eta_{e}^{pc}/\eta_0 \approx 2.2 \) (neglecting \( \nu_{ee} \) effects, which would make the results a factor of 0.6 smaller), \( \delta_e \approx 10^{-3} \text{ m}, n_{max} \approx 11, \) and \( \lambda_e \approx 300 \text{ m} > \pi R_0 q n_{max} \approx 140 \text{ m}, \) so that \( L \approx \pi R_0 q n_{max}, M = n_{max} \approx 11, \) and the estimated \( \chi_e^{pc} \) is 2.5 m$^2$/s $\sim \chi_e^{exp}$. Since this \( \chi_e^{pc} \propto n_e^{1/4}/(q T_e^3)^{1/2} \propto T_e(r)^{-3/2} \) (again neglecting \( \nu_{ee} \) effects), the radial dependence of \( \chi_e^{pc} \) increases strongly as \( T_e \) decreases with increasing \( r \), in qualitative agreement with inferences from experiments.

In high density, more collisional plasmas where \( L = \lambda_e, M = \lambda_e/\pi R_q \gg 1 \) yields

\[
\chi_e^{pc} \approx \frac{3}{2} \frac{\eta_{e}^{pc}}{\eta_0} \frac{v_T e^2}{\pi R_q \omega_p^2}, \quad \pi R_q < \lambda_e < \pi R_q n_{max}, \tag{52}
\]

which will be referred to as the collisional paleoclassical regime. In typical high density toroidal plasmas \( Z_{eff} \approx 1 \) and \( \nu_{ee} > 1; \) for such plasmas \( (3/2)(\eta_{e}^{pc}/\eta_0) \approx (1.5)(0.51). \)
Paleoclassical electron heat transport

Thus, the collisional $\chi^{\text{pc}}_e$ implies an overall electron energy confinement time $\tau_{Ee} \sim a^2/4\chi^{\text{pc}}_e \approx 0.27 (n_e/10^{20} \text{m}^{-3}) a^2 R q_0 T_e/500 \text{eV}^{-1/2}$ s, which approximately reproduces (in both magnitude and scaling for the highest performance pellet-fueled Alactor C plasmas [15] that had $a = 0.165$ m and $R_q = 0.64$ m) the “neo-Alcator scaling” deduced empirically primarily from ohmically-heated tokamak plasma data in the 1970s and early 1980s [16]: $\tau_{Ee}^A \approx 0.07 n_e a R_q^2$. Because both ohmic heating and paleoclassical transport are proportional to the neoclassical parallel resistivity, in ohmically heated tokamak plasmas the electron power balance $\eta^{\text{pc}}_e J^2 = \partial (Q^{\text{pc}}_e \cdot \nabla V)/\partial V$ yields a scaling $\beta_p \equiv \tilde{P}/[B_p(a)^2/2\mu_0] \sim 1/M$. This is in reasonable accord with early ohmic tokamak experimental data [17] since for the more collisional plasmas of the early 1970s $\beta_p \sim 1/M \sim \pi R q_0/\lambda_e \sim 0.2–1$ (or $\propto n_e/I$ for $T_e$ approximately constant). For present-day, higher $T_e$, more collisionless, ohmically-heated tokamak plasmas $\beta_p \sim 1/n_\text{max} \sim 0.1$.

In the closed field line region near but inside the magnetic separatrix region of divertor plasmas where $q$ and $q'$ become very large, one can have $\lambda_e \lesssim \pi R q$. In this near-separatrix (subscript s) region, the paleoclassical electron heat diffusivity is [2]

$$\chi^{\text{pc}}_{es} \sim \frac{3 \eta^{\text{nc}}_e}{2 \mu_0} \left(1 + \frac{\eta^{\text{nc}}_e \lambda_e}{\eta^{\text{Sp}}_e \pi R q}\right), \quad \pi R < \lambda_e < \pi R q \max \{1, n_\text{max}\}. \tag{53}$$

For $\lambda_e/\pi R q > (\eta^{\text{Sp}}_e/\eta^{\text{nc}}_e) \sim 1$, this yields the collisional $\chi^{\text{pc}}_e$ in (52). In the opposite limit one obtains a smaller $\chi^{\text{pc}}_{es} \simeq (3/2)(\eta^{\text{Sp}}_e/\mu_0) \simeq Z_{\text{eff}} [100/T_e(\text{eV})]^{3/2} \text{m}^2/\text{s}$. There are some experimental indications in DIII-D [18,19] that within about 2 cm of the separatrix $\nabla T_e$ is significantly larger, which implies $\chi^{\text{pc}}_{es}$ is reduced there. The paleoclassical model predicts that the maximum $\nabla T_e$ should occur where the two terms in (53) become comparable: $q(\rho^s) \sim (\lambda_e/\pi R) (\eta^{\text{nc}}_e/\eta^{\text{Sp}}_e) \sim 5–10$. The indicated $\rho^s \sim 0.95–0.97$ is in reasonable agreement with where the maximum $\nabla T_e$ occurs in the experiments [18,19].

The paleoclassical model applies to all types of axisymmetric toroidal plasmas in regions where $c^2 B_p^2/B_t^2 \ll 1$. For $R_0 \simeq 1 \text{ m STs with } T_e \sim 1 \text{ keV and } n_e \sim 3 \times 10^{19} \text{ m}^{-3}$, the prediction at $r/a \sim 0.5$ is $\chi^{\text{pc}}_e \sim 5–10 \text{ m}^2/\text{s}$, which, in reasonable agreement with experimental results [20,21], is large because for STs $q' \ll 1$ is small and $\eta^{\text{nc}}_e/\eta_0 \gtrsim 3$ is large in the plasma confinement region ($r/a \sim 0.5$). For quiescent RFP plasmas in the Madison Symmetric Torus (MST) Pulsed Poloidal Current Drive (PPCD) experiments [22,23], at $r/a \sim 0.3–0.5$ one obtains $\chi^{\text{pc}}_e \sim 5–10 \text{ m}^2/\text{s}$ (large because $q < 0.2$ and $|q'| \lesssim 0.2$ are small), which is close to the effective $\chi_e$’s inferred from global ($\chi^{\text{exp}}_e \equiv a^2/4\tau_E \sim 7.5 \text{ m}^2/\text{s}$ [22]) and local ($\chi^{\text{exp}}_e \sim 10–30 \text{ m}^2/\text{s}$ [23]) measurements. In quasi-symmetric stellarator plasmas there would be no paleoclassical transport if there is no flux-surface-average parallel current $\langle J \cdot B \rangle$; however, net flux-surface-average parallel currents in quasi-symmetric stellarators would apparently induce a $\chi^{\text{cell}}_e \simeq [\kappa_j/(\kappa_v + \kappa_j)]\chi^{\text{pc}}_e$ because while the field lines due to the current ($\kappa_j$) would diffuse radially, those due to the vacuum fields ($\kappa_v$) would not — as can be deduced from (28) and (29) for a $\delta \psi$ composed of diffusing and spatially constant parts.
As indicated by (44), (46), and (50), the predicted $\chi_{e}^{pc}$ is much smaller for the “short” helical field lines [see (2) and discussion thereafter] in the vicinity of low order rational surfaces with $q^o = m^0/n^0$: $\chi_{e}^{pc} \sim (3/2) (n^o + 1) \eta_{e}^{pc} / \mu_0$. The estimated width of the low $\chi_{e}^{pc}$ “transport barriers” is $2 \delta x^o$ for $q' \neq 0$, or, if $q$ is near a minimum at the rational surface, $2 \delta x_{min}^o$: the distances $\delta x^o$ and $\delta x_{min}^o$ are defined in (47) and (48), respectively. These barrier widths can be compared to some key tokamak results. First, as experiments in RTP [24] slowly moved highly localized electron cyclotron heating (ECH) radially outward, a “stair-step” reduction in the central $T_e$ was observed as the ECH passed low order rational surfaces. It was thus inferred [24] that transport barriers existed with up to a factor of 10 reduction in $\chi_{e}$ over relative (to $a$) barrier widths of order 0.04 (0.1 for $q = 1/1$). For RTP parameters $2 \delta x^o \sim 0.06-0.12$ (0.17 for $q = 1/1$). Next, jumps in $T_e$ (over radial widths $\sim 0.2$) have been observed in evolving DIII-D L-mode plasmas [25] as an off-axis minimum in $q(\rho, t)$ passes through low order rational surfaces. For the DIII-D parameters, $2 \delta x_{min}^o$ gives a similar estimate ($\sim 0.3$) for the transport barrier width. Finally, a large, localized $\nabla T_e$, which implies an electron internal transport barrier, was created in the pioneering JT-60U experiments [26]; presuming $q$ was near a minimum [27] at $q_{min} = 3$, the width predicted by paleoclassical theory is 0.14, which is close to the experimentally inferred barrier width of about 0.11.

Note that the paleoclassical electron heat flux in (49) is not in a normal (diffusive) Fourier heat flux law form (i.e., $q_e = -\kappa_e \nabla T_e \equiv -n_e \chi_e \nabla T_e$). Rather, it can be written in general as:

$$\langle Q_e^{pc} \cdot \nabla V \rangle = -V' n_e \chi_e^{pc} \frac{\partial T_e}{\partial \rho} - \langle q_e^{pl} \cdot \nabla V \rangle,$$

$$\langle q_e^{pl} \cdot \nabla V \rangle \equiv T_e \frac{\partial}{\partial \rho} (V' n_e \chi_e^{pc}), \quad \text{heat pinch.}$$

The electron heat pinch heat flux $\langle q_e^{pl} \cdot \nabla V \rangle$ is usually positive (inward) and increases slightly with $\rho$, in qualitative agreement with JET [12] and recent [28] tokamak experimental results. Also, a heat pinch effect implies [12] a “power balance” $\chi_e (\chi_e^{pb})$, which is defined as the net electron heat flux divided by $-n_e \nabla T_e$, that is less than $\chi_e^{pc}$. Alternatively, in qualitative agreement with experimental data from many tokamaks [29], $\langle Q_e^{pc} \cdot \nabla V \rangle$ can be written in the form of a heat flux proportional to the degree to which the electron temperature gradient exceeds a critical magnitude defined by

$$\left. \frac{1}{L_{Te} \text{crit}} \right| \equiv -\langle \hat{e}_e \cdot \nabla \ln T_e \rangle \approx \frac{\partial}{\partial r} \ln (V' n_e \chi_e^{pc}).$$

If this paleoclassical critical gradient is approximately constant over the confinement region, it would agree with the experimental observations that $\nabla \ln T_e$ in the “confinement region” (0.3 $\lesssim \rho \lesssim$ 0.8) is nearly constant [30] and usually close to its critical value.
6. Paleoclassical model applicability, tests

Paleoclassical electron heat transport is based on the primitive Coulomb collision processes of parallel electron heat conduction and plasma resistivity leading to magnetic field diffusion in low collisionality toroidal plasmas. Thus, it is an “irreducible, ubiquitous” transport process, just as classical and neoclassical transport [5,6] are. Hence, it sets a “base” level of electron heat transport in current-carrying axisymmetric toroidal plasmas.

Plasma turbulence produced by microinstabilities might induce additional transport. Since \( D_\eta \propto \eta \propto 1/T_e^{3/2} \), the paleoclassical electron heat diffusion coefficient \( \chi_{\text{pc}}^e \) also decreases with increasing electron temperature as \( 1/T_e^{3/2} \) (neglecting \( \nu_{se} \) effects which weaken the inverse \( T_e \) scaling); for \( n_{\text{max}} < 10 \) and \( \eta_{||}^{\text{pc}} / \eta_0 < 2 \), \( \chi_{\text{pc}}^e \) becomes less than 1 \( m^2/s \) for \( T_e < 2 \) keV. In contrast, since microturbulence-induced diffusion tends to be a multiple of the gyro-Bohm coefficient \( \chi_{\text{gb}}^e \equiv (\theta_0/a)(T_e/eB) \propto T_e^{3/2} \), it tends to increase with \( T_e \). Thus, if plasma microinstabilities induce \( \chi_{\text{micro}}^{\text{turb}} > 1 \) \( m^2/s \) for \( T_e < 2 \) keV, they would become the dominant transport mechanisms as electron temperatures increase above a few keV. However, when microinstabilities are stabilized or the transport they induce is strongly reduced (e.g., by sheared \( \mathbf{E} \times \mathbf{B} \) flows or in transport barriers), the paleoclassical electron heat transport would still set the base \( T_e \) transport level, even at high \( T_e \). Also, since \( \chi_{\text{pc}}^e / \chi_{\text{gb}}^e \propto 1/T_e^{3} \), paleoclassical radial electron heat transport is likely to be dominant toward and in the cooler edge of toroidal plasmas — perhaps wherever \( T_e < 1 \) keV.

Because the physical processes underlying paleoclassical and microturbulence-induced electron heat transport are so different, they may be mostly independent of each other; hence, the transport they induce might be just additive. Experimentally, the parallel Ohm’s law seems to be well represented by the neoclassical predictions for both axisymmetric poloidal [31] and helically-symmetric [32] magnetic field evolution. Theoretically, paleoclassical transport processes would apparently not be directly affected by microturbulence since: 1) microturbulence usually does not significantly affect the parallel Ohm’s law [33,34] because its parallel wavenumbers and hence momentum transfer are small; and 2) the parallel correlation length for magnetic microturbulence usually exceeds the relevant paleoclassical length \( L \).

To distinguish between various types of radial electron heat transport processes it would be useful if experimentalists could begin referencing their inferred \( \chi_{\text{gb}}^e \) values to the characteristic diffusivities for the various possible mechanisms for \( \chi_{\text{e}}^e \):

\[
D_{\eta_0} \equiv \frac{\eta_0}{\mu_0} \simeq 0.05 \frac{Z}{[T_e(\text{keV})]^{3/2}} \frac{m^2}{s}, \tag{57}
\]

\[
\chi_{\text{gb}}^e \equiv \frac{\theta_0}{a} \frac{T_e}{eB} \simeq 3 \frac{[T_e(\text{keV})]^{3/2}A_1^{1/2}}{a(m)[B(T)]^{3/2}} \frac{m^2}{s}, \tag{58}
\]

\[
\chi_{\text{pc}}^e \equiv \frac{\theta_0}{a} \frac{T_e}{eB} \simeq 0.1 \frac{[T_e(\text{keV})]^{3/2}}{a(m)[B(T)]^{3/2}} \frac{m^2}{s}. \tag{59}
\]

The paleoclassical \( \chi_{\text{pc}}^e \) is predicted to be a multiple (typically \(~ 10-30 \gg 1\)) of the
magnetic field diffusivity $D_\eta$ and hence should be referenced to it, or for simplicity the reference diffusivity $D_{\eta_{0}}$. Ion-temperature-gradient (ITG) driven and other drift-wave-type (e.g., trapped electron) microturbulence typically lead to predictions for $\chi_e$ that are a multiple (typically $\sim 1\text{--}3$) of the ion sound gyro-Bohm diffusivity $\chi_{i}^{gB}$. Finally, electron-temperature-gradient (ETG) microturbulence [35] leads to $\chi_e$ predictions that are multiples (perhaps as large as 60 [36], but maybe only of order 3 [37]) of the electron gyro-Bohm diffusivity $\chi_{e}^{gB}$. Hence, determining how large the experimentally inferred $\chi_e$’s are relative to the characteristic diffusion coefficients $D_{\eta_{0}}$, $\chi_{i}^{gB}$ and $\chi_{e}^{gB}$ given in (57)--(59) may yield clues about the type of physical processes (paleoclassical, drift-wave, or ETG type) responsible for radial electron heat transport — particularly for low power density, low electron density, and/or $T_e > T_i$ regimes where the electron and ion heat transport channels are weakly coupled and the electron transport processes can be somewhat independently explored.

Transport scalings for magnetically confined plasmas plasma are often sought in terms of dimensionless physical variables. The local diffusivities are usually scaled relative to the Bohm diffusivity $T/eB$, as in (58), (59). The global energy confinement time ($\tau_E \sim a^2/4\chi$) is usually scaled relative to the gyrofrequency $\omega_e \equiv qB/m$. Thus, in terms of physically relevant dimensionless variables for magnetized toroidal plasmas such as a normalized gyroradius $q_* \equiv q/a$, relative pressure $\beta \equiv P/(B^2/2\mu_0)$, and neoclassical collisionality $\nu_{*e}$ the scalings are of the form $\chi/\chi^{gB} \sim h(q_*, \beta, \nu_{*e})$ and $\omega_e\tau \sim f(q_*, \beta, \nu_{*e})$. While these are the relevant scalings and dimensionless physical variables for transport induced by drift-wave and ETG microturbulence, they are not appropriate for paleoclassical transport.

The natural parameters against which the paleoclassical local diffusivity $\chi_{e}^{pc}$ and global electron energy confinement time $\tau_{Ee} \sim a^2/4\chi_{e}^{pc}$ should be scaled are the magnetic field diffusivity $D_\eta$ and skin time $\tau_{sk} \sim a^2/6D_\eta$ induced by the reference plasma resistivity $\eta_{0}/\mu_0$. Thus, in terms of the paleoclassical-relevant physical parameters of the normalized em skin depth $\delta_e$ and neoclassical electron collisonality $\nu_{*e}$, their natural scalings are:

$$\frac{\chi_{e}^{pc}}{\eta_{0}/\mu_0} \equiv \frac{\chi_{e}^{pc}}{\nu_{e}\delta_e^2} \sim f_\chi(\delta_e, \nu_{*e}) g_\chi(q', q, \epsilon),$$

$$\frac{\tau_{Ee}^{pc}}{\tau_{sk}} \sim \frac{1}{f_\chi^{-1}(\delta_e, \nu_{*e}) g_\chi^{-1}(q', q, \epsilon)},$$

in which $f_\chi$ and $g_\chi$ are functions of the relevant dimensionless physical and geometric variables, respectively; $f_\chi^{-1}$ and $g_\chi^{-1}$ are appropriate spatial averages (e.g., from [12]) of the reciprocals of these functions. In the normally applicable collisionless paleoclassical regime (51) one has $f_\chi \sim 1/\sqrt{\delta_e}$ and $g_\chi \sim 1/\sqrt{q'}$ for $\nu_{*e} << 1$; a $\nu_{*e}$ dependence of $f_\chi$ and a dependence of $g_\chi$ on $q$ and $\epsilon$ can develop both from the neoclassical parallel resistivity (14) or as one moves into the collisional paleoclassical regime (52), usually toward the cooler plasma edge.

In terms of the natural variables for paleoclassical transport, in the normal collisionless paleoclassical regime the local electron heat diffusivity and electron energy
confinement time depend on only one dimensionless physical parameter \( \delta_e \) — via the dependence of \( f_\lambda \) on it. Perhaps this explains why electron heat transport is so ubiquitous and does not vary much between ohmically-heated toroidal plasmas in which the plasma resistivity plays a key role in both the ohmic heating and the paleoclassical transport. Finally, it should be noted that, as with neoclassical transport, the paleoclassical scaling laws in (60) and (61) probably cannot be probed by step increases in auxiliary electron heating power that successively increase \( T_e \) since in such a scenario \( D_{\eta_0} \sim 1/T_e^{3/2} \) decreases whereas microturbulence-induced transport which scales as \( \chi_{e}^{PB} \sim T_e^{3/2} \) increases and is likely to become the limiting transport mechanism.

Since the magnetic field does not appear explicitly in any of the parameters in (60) and (61), one might wonder how they could possibly apply to “magnetically-confined” plasmas? However, the magnetic field is implicitly involved since \( D_{\eta_0} \) represents the radial diffusion of the magnetic field and \( q \) represents the ratio of the poloidal to toroidal magnetic field. Thus, paleoclassical transport processes do involve the magnetic field, albeit implicitly, and hence are applicable to magnetically-confined toroidal plasmas.

The present paleoclassical analysis is limited [2] to near-equilibrium situations where \( D_\eta \gg |d\psi/dt (q_\ast/q'\psi')| \) — so magnetic field lines diffuse radially more than transient magnetic flux changes move them in radius. More physically, this criterion implies that at all radii in the plasma the transient-induced changes in the parallel electric field \( \langle \delta E \cdot B \rangle \) are small compared to the equilibrium electric field \( \langle E_0 \cdot B \rangle \) needed to drive the flux-surface-averaged parallel current \( J_0 \) in equilibrium:

\[
\langle \delta E \cdot B \rangle \ll \langle E_0 \cdot B \rangle \equiv \eta_\parallel^{pe} \langle J_0 \cdot B \rangle.
\] (62)

Many types of transient transport experiments apparently violate this criterion and hence cannot be analyzed with the present quasi-stationary paleoclassical model.

With the preceding limitations in mind, various types of local tests of the paleoclassical model are possible and suggested for quasi-stationary plasmas:

A) Electron Heat Diffusivity: 1) Is \( \chi_e^{pc} \) close to the experimentally inferred \( \chi_e^{pb} \) in magnitude?; and 2) Do their radial profiles agree? It would be helpful if \( \chi_e^{pc} \) could be routinely evaluated and displayed in plots of \( \chi_e^{pb} \) versus \( \rho \). (As a first step, the effects of low order rational surfaces in \( \chi_e^{pc} \) could be neglected for simplicity.)

B) Electron Internal Transport Barriers (eITBs): 1) Do eITBs occur at all low order rational surfaces where \( q^\circ \equiv m^\circ/n^\circ \) with \( n^\circ = 1, 2, \ldots \); 2) Do the widest eITBs occur around minima in \( q \) about \( q^\circ \)? 2) Do the radial barrier widths agree with \( 2\delta x^0 \) from (47) [or \( 2\delta x_{\min}^0 \) from (48) if \( q^\circ \) is at a minimum in \( q \)]?; 3) Is \( \chi_e^{pc} \) approximately constant within the eITB but then increasing rapidly a few \( \delta_e \) from the nearest \( n_{\max} \) rational surface?; and 4) Are the barrier depths in agreement with the paleoclassical prediction of \( \chi_e^{pc} \) (in eITB)/\( \chi_e^{pc} \) (outside eITB) \( \sim (n^\circ + 1)/n_{\max} \sim 1/5 \)?

C) Electron Heat Flux: 1) In cases where the paleoclassical electron heat transport is likely to be dominant \( (T_e \lesssim 1 \text{ keV}) \), is the experimental heat flux data well represented by Eq. (54) — i.e., with a heat pinch or minimum temperature gradient
form; 2) If so, does the magnitude of the electron heat pinch agree with (55) — or the minimum $T_e$ gradient agree with (56)?; and 3) Is the $1/L_{Te|\text{crit}}$ in (56) nearly constant over the main confinement region of $0.3 \lesssim \rho \lesssim 0.8$?

D) Non-tokamak Experiments: 1) Is $\chi_{e|\text{PC}}$ large in low aspect ratio ST plasmas primarily because of large $\eta^0_{n|e}$ or small $q'$ or a combination of them?; 2) Is $\chi_{e|\text{PC}}$ large in quiescent RFP plasmas primarily because $q$ and $q'$ are small?; and 3) Is $\chi_{e|\text{pb}} \sim [\ell_J/(\ell_J + \ell_J)]\chi_{e|\text{PC}}$ in quasi-symmetric, current-carrying stellarator plasmas?

E) Fundamental Physics Tests: 1) Are both the poloidal and helical magnetic field diffusion governed by the neoclassical parallel resistivity, as initial indications seem to suggest [31,32]; 2) Does the spatial structure of the electron temperature develop helical distortions in the vicinity of rational surfaces as indicated by (39) which implies [2] $\hat{T}_e(\alpha, x) \sim M x e^{i\alpha a} dT_e/d\rho$ for $|x| \lesssim \delta x(n)/2$?; 3) In a linear device (or just inside a magnetic separatrix) is $\chi_e \simeq (3/2)D_{\eta|e}$?; and 4) Most generally, is $\chi_e \sim (L/\pi R q)D_{\eta|e}$ with $L = \min\{\ell_{\text{max}}, \lambda_e, n^o\}$ for each of the limiting cases for $L$?

As indicated in the discussion in the previous section, many “back-of-the-envelope” type tests have already been carried out with mostly encouraging results. What’s needed now are more precise and detailed tests by experimentalists and transport modelers. In making these various tests it should be kept in mind that because the paleoclassical results were obtained by a large $n$ asymptotic analysis and the characteristic lengths in $L$ have been determined only approximately, $M$ (and hence all $M$-dependent results) should be interpreted as scaling results. Thus, numerical coefficients of the order of up to a factor of say 2 (up or down) should be allowed for in the various components of $L$, and hence in $M$ and $\chi_{e|\text{PC}}$.

7. Summary

Equations (49)–(54) are the main paleoclassical results. As indicated in Section 5, they provide interpretations for many features of “anomalous” electron heat transport. Because the results were obtained by a large $n$ asymptotic analysis and the characteristic lengths in $L$ have been determined only approximately, $M$ (and hence all $M$-dependent results) should be interpreted as scaling results. More detailed studies could introduce numerical coefficients of order unity in $L$, $M$, and $\chi_{e|\text{PC}}$.

Paleoclassical processes provide a ubiquitous, irreducible and hence base level of radial electron heat transport. However, they probably do not provide the limiting transport with auxiliary heating. Microturbulence-induced transport adds to the paleoclassical levels and likely becomes dominant at high $T_e$ ($\gtrsim 2$ kev?).

References

Paleoclassical electron heat transport


[18] Osborne T.H. 2004 (private communication)


[21] LeBlanc B.P. *et al* 2004 Nucl. Fusion **44** 513


[25] Austin M.E. 2004 (private communication), oral talk at 2001 DPP-APS meeting, Long Beach, CA


[30] Ryter F. *et al* 2001 Plasma Phys. Control. Fusion **43** A323 Fig. 2


[37] Lin Z. *et al* 2004 Paper TH/8-4 at 20th IAEA Fusion Energy Conference, 1-6 November Vilamoura, Portugal