Mathematical Models
Underlying Governing Equations, Principles and Variables

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Electrodynamics


(Modified) Tonti Diagram

- Unknown primary variable
- Unknown secondary variable; computed as a post-processing step
- Known quantity
- Relationship enforced in a ‘strong’ sense inside the domain
- Relationship enforced in a ‘strong’ sense on the boundary
- Relationship enforced in a ‘weak’ sense inside the domain
- Relationship enforced in a ‘weak’ sense on the boundary
Example: Three-dimensional transient heat conduction for an isotropic material.

Primary Unknown: $T(x, y, z, t)$

Kinetic law: $G(x, y, z, t) = \left( G_x, G_y, G_z \right) = \left( \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right) = \nabla T$

Material law: $Q(x, y, z, t) = \left( Q_x, Q_y, Q_z \right) = -\left( kG_x, kG_y, kG_z \right)$

Flux law: $q_f = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} = \nabla Q$

Inertial law: $q_i = \rho c \frac{\partial T}{\partial t}$

Balance principle: $q(x, y, z, t) = q_i + q_f$

Differential equation: $\Rightarrow q = \nabla \cdot \left( -k \nabla T \right) + \rho c \frac{\partial T}{\partial t}$
Example: One-dimensional steady-state heat conduction for an isotropic material.

Primary Unknown: $T(x)$
Kinetic law: $G(x) = (G_x) = \left( \frac{dT}{dx} \right) = \nabla T$
Material law: $Q(x) = (Q_x) = -(kG_x)$
Flux law: $q_f = \frac{dQ_x}{dx} = \nabla Q$
Inertial law: $q_i = 0$
Balance principle: $q(x) = q_i + q_f$
Differential equation: $\frac{d}{dx} \left( -k \frac{dT}{dx} \right) = q(x)$