Chapter 1

Collective Plasma Phenomena

The properties of a medium are determined by the microscopic processes in it. In a plasma the microscopic processes are dominated by collective, rather than binary, charged particle interactions — at least for sufficiently long length and time scales.

When two charged particles are very close together they interact through their Coulomb electric fields as isolated, individual particles. However, as the distance between the two particles increases beyond the mean particle separation distance ($n^{-1/3}$, in which n is the charged particle density), they interact simultaneously with many nearby charged particles. This produces a *collective* interaction. In this regime the Coulomb force from any given charged particle causes all the nearby charges to move, thereby electrically polarizing the medium. In turn, the nearby charges move collectively to reduce or "shield out" the electric field due to any one charged particle, which in the absence of the shielding decreases as the inverse square of the distance from the particle. In equilibrium the resultant "cloud" of polarization charge density around a charged particle has a collectively determined scale length — the *Debye shielding* length — beyond which the electric field due to any given charged particle is collectively shielded out. That is, the "long range force" of the Coulomb electric field is actually limited to a distance of order the Debye length in a plasma.

On length scales longer than the Debye length a plasma responds collectively to a given charge, charge perturbation, or imposed electric field. The Debye shielding distance is the maximum scale length over which a plasma can depart significantly from charge neutrality. Thus, plasmas, which must be larger than a Debye length in size, are often said to be *quasineutral* — on average electrically neutral for scale lengths longer than a Debye length, but dominated by the charge distribution of the discrete charged particles within a Debye length.

Most plasmas are larger than the Debye shielding distance and hence are not dominated by boundary effects. However, boundary effects become important within a few Debye lengths of a material limiter or wall. This boundary region, which is called the *plasma sheath* region, is not quasineutral. Material probes inserted into plasmas, which are called *Langmuir probes* after their developer (in the 1920s) Irving Langmuir, can be biased (relative to the plasma) and draw currents through their surrounding plasma sheath region. Analysis of the current-voltage characteristics of such probes can be used to determine the plasma density and electron temperature.

If the charge density in a quasineutral plasma is perturbed, this induces a change in the electric field and in the polarization of the plasma. The small but finite inertia of the charged particles in the plasma cause it to respond collectively — with Debye shielding, and oscillations or waves. When the characteristic frequency of the perturbation is low enough, both the electrons and the ions can move rapidly compared to the perturbation and their responses are *adiabatic*. Then, we obtain the Debye shielding effect discussed in the preceding paragraphs.

As the characteristic frequency of the perturbations increases, the inertia of the charged particles becomes important. When the perturbation frequency exceeds the relevant inertial frequency, we obtain an *inertial* rather than adiabatic response. Because the ions are much more massive than electrons (the proton mass is 1836 times that of an electron — see Section A.8 in Appendix A), the characteristic inertial frequency is usually much lower for ions than for electrons in a plasma. For intermediate frequencies — between the characteristic electron and ion inertial frequencies — electrons respond adiabatically but ions have an inertial response, and the overall plasma responds to perturbations via *ion acoustic waves* that are analogous to sound waves in a neutral fluid. For high frequencies — above the electron and ion inertial frequencies — both electrons and ions exhibit inertial responses. Then, the plasma responds by oscillating at a collectively determined frequency called the *plasma frequency*. Such "space charge" oscillations are sometimes called Langmuir oscillations after Irving Langmuir who first investigated them in the 1920s.

In this chapter we derive the fundamental collective processes in a plasma: Debye shielding, plasma sheath, plasma oscillations, and ion acoustic waves. For simplicity, in this chapter we consider only unmagnetized plasmas — ones in which there is no equilibrium magnetic field permeating the plasma. At the end of the chapter the length and time scales associated with these fundamental collective processes are used to precisely define the conditions required for being in the plasma state. Discussions of applications of these fundamental concepts to various basic plasma phenomena are interspersed throughout the chapter and in the problems at the end of the chapter.

1.1 Adiabatic Response; Debye Shielding

To derive the Debye shielding length and illustrate its physical significance, we consider the electrostatic potential ϕ around a single, "test" charged particle in a plasma. The charged particles in the plasma will be considered to be "free" charges in a vacuum. Thus, the electrostatic potential in the plasma can be determined from

$$\nabla \cdot \mathbf{E} = -\nabla^2 \phi = \rho_q / \epsilon_0, \quad \text{Poisson equation,} \tag{1.1}$$

which results from writing the electric field $\mathbf{E}$ in terms of the electrostatic potential, $\mathbf{E} = -\nabla \phi$, in Gauss's law — see (**??**) and (**??**) Section A.2. The charge density ρ_q is composed of two parts: that due to the test charge being considered and that due to the *polarization* of the plasma caused by the effect of the test particle on the other charged particles in the plasma. Considering the test particle of charge q_t to be a point charge located at the spatial position $\mathbf{x}_t$ and hence representable[1] by $\delta(\mathbf{x} - \mathbf{x}_t)$, the charge density can thus be written as

$$\rho_q(\mathbf{x}) = q_t \, \delta(\mathbf{x} - \mathbf{x}_t) + \rho_{\text{pol}}(\mathbf{x}) \tag{1.2}$$

in which ρ_{pol} is the polarization charge density.

The polarization charge density results from the responses of the other charged particles in the plasma to the Coulomb electric field of the test charge. For slow processes (compared to the inertial time scales to be defined more precisely in Section 1.4 below), the responses are *adiabatic.* Then, the density of charged particles (electrons or ions) with charge[2] q and temperature T in the presence of an electrostatic potential $\phi(\mathbf{x})$ is given by [see (**??**) in Section A.3]

$$\boxed{n(\mathbf{x}) = n_0 e^{-q\phi(\mathbf{x})/T}, \quad \text{Boltzmann relation (adiabatic response),}} \tag{1.3}$$

where n_0 is the average or equilibrium density of these charged particles in the absence of the potential. The potential energy $q\phi$ of our test particle will be small compared to its thermal energy, except perhaps quite close to the test particle. Thus, we expand (1.3) assuming $q\phi/T << 1$:

$$n \simeq n_0 (1 - \frac{q\phi}{T} + \frac{1}{2} \frac{q^2 \phi^2}{T^2} \cdots), \quad \text{perturbed adiabatic response.} \tag{1.4}$$

The validity of this expansion will be checked *a posteriori* — at the end of this section. To obtain the desired polarization charge density ρ_{pol} caused by the effect of the potential ϕ on all the charged particles in the plasma, we multiply (1.4) by the charge q for each species s ($s = e, i$ for electrons, ions) of charged particles and sum over the species to obtain

$$\rho_{\text{pol}} \equiv \sum_s n_s q_s = -\sum_s \frac{n_{0s} q_s^2}{T_s} \phi \left[1 + \mathcal{O}\left(\frac{q_s \phi}{T_s} \right) \right] \tag{1.5}$$

in which the "big oh" $\mathcal{O}$ indicates the order of the next term in the series expansion. In obtaining this result we have used the fact that on average a

[1] See Section B.2 in Appendix B for a discussion of the Dirac delta function $\delta(\mathbf{x})$.

[2] Throughout this book q will represent the signed charge of a given plasma particle and $e \simeq 1.602 \times 10^{-19}$ coulomb will represent the magnitude of the elementary charge. Thus, for electrons we have $q_e = -e$, while for ions of charge Z_i we have $q_i = Z_i e$.

plasma is electrically *quasineutral:*

$$\boxed{\sum_s n_{0s} q_s = 0, \quad \text{quasineutrality condition.}} \tag{1.6}$$

Retaining only the lowest order, linear polarization charge density response in (1.5), substituting it into (1.2), and using the resultant total charge density in the Poisson equation (1.1), we obtain

$$\left(-\nabla^2 + \frac{1}{\lambda_D^2}\right)\phi = \frac{q_t}{\epsilon_0}\,\delta(\mathbf{x}-\mathbf{x}_t) \tag{1.7}$$

in which the $1/\lambda_D^2$ term is caused by the polarizability of the plasma. Here, λ_D is the Debye shielding length:

$$\frac{1}{\lambda_D^2} \equiv \sum_s \frac{1}{\lambda_{Ds}^2} \equiv \sum_s \frac{n_{0s} q_s^2}{\epsilon_0 T_s} = \frac{1}{\lambda_{De}^2} + \frac{1}{\lambda_{Di}^2} = \frac{n_{0e} e^2}{\epsilon_0 T_e} + \frac{n_{0i} Z_i^2 e^2}{\epsilon_0 T_i}, \quad \text{plasma Debye length.} \tag{1.8}$$

In the last expressions we have assumed a plasma composed of electrons with density n_{0e} and only one species of ions with charge $Z_i e$ and density n_{0i}. Note that for comparable electron and ion temperatures the electron and ion Debye lengths are comparable. The overall plasma Debye length is obtained from the sum of the inverse squares of the Debye lengths of the various species of charged particles in the plasma. For a plasma composed of electrons and protons, which we will call an electron-proton plasma, the lower temperature component will give the dominant contribution to the overall plasma Debye length. Numerically, the electron Debye length is given in SI (mks) units by

$$\boxed{\lambda_{De} \equiv \sqrt{\frac{\epsilon_0 T_e}{n_e e^2}} \simeq 7434\sqrt{\frac{T_e(\text{eV})}{n_e(\text{m}^{-3})}}\ \text{m}, \quad \text{electron Debye length.}} \tag{1.9}$$

The general solution of (1.7) in an infinite, homogeneous three-dimensional plasma geometry[3] is[4]

$$\phi_t(\mathbf{x}) = \frac{q_t\, e^{-|\mathbf{x}-\mathbf{x}_t|/\lambda_D}}{\{4\pi\epsilon_0\}\,|\mathbf{x}-\mathbf{x}_t|} = \frac{q_t\, e^{-r/\lambda_D}}{\{4\pi\epsilon_0\}\, r}, \quad \text{potential around a test particle.} \tag{1.10}$$

Here, the subscript t indicates this is the particular solution for the potential around a test charge q_t in a plasma. That this is the solution can be verified by substituting it into (1.7), noting that $(-\nabla^2 + 1/\lambda_D^2)\phi_t = 0$ everywhere except where $r \equiv |\mathbf{x}-\mathbf{x}_t| \to 0$ and there $\lim_{r\to 0}\int d^3x\ \nabla^2\phi = \lim_{r\to 0}\oiint d\mathbf{S}\cdot\boldsymbol{\nabla}\phi =$

[3]For one- and two-dimensional geometries see Problems 1.4 and 1.5.

[4]Here and throughout this book we write the mks factor $\{4\pi\epsilon_0\}$ in braces; eliminating this factor yields the corresponding cgs (Gaussian) forms for electrostatic response formulas.

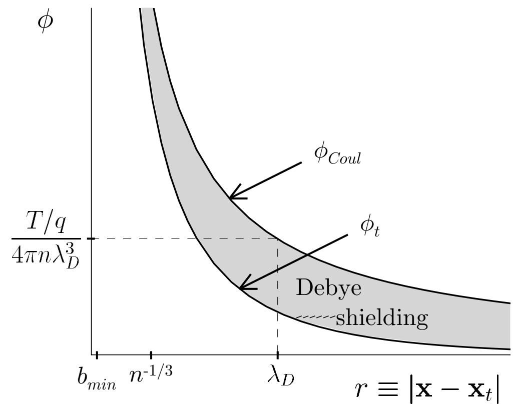

Figure 1.1: Potential ϕ_t around a test particle of charge q_t in a plasma and Coulomb potential $\phi_{\rm Coul}$, both as a function of radial distance from the test particle. The shaded region represents the Debye shielding effect. The characteristic distances are: λ_D, Debye shielding distance; $n_e^{-1/3}$, mean electron separation distance; $b_{\rm min}^{\rm cl} = q^2/(\{4\pi\epsilon_0\}T)$, classical distance of "closest approach" where the $e\phi/T << 1$ approximation breaks down.

$-q_t/\epsilon_0$. The solution given in (1.10) is also the Green function for the equation $(-\nabla^2 + 1/\lambda_D^2)\phi = \rho_{\rm free}/\epsilon_0$ — see Problem 1.6.

The potential around a test charge in a plasma, (1.10), is graphed in Fig. 1.1. Close to the test particle (i.e., for $r \equiv |\mathbf{x} - \mathbf{x}_t| << \lambda_D$), the potential is simply the "bare" Coulomb potential $\phi_{\rm Coul} = q_t/\left(\{4\pi\epsilon_0\}\,|\mathbf{x} - \mathbf{x}_t|\right)$ around the test charge q_t. For separation distances of order the Debye length λ_D, the exponential factor in (1.10) becomes significant. For separations large compared to the Debye length the potential ϕ_t becomes exponentially small and hence is "shielded out" by the polarization cloud surrounding the test charge. Overall, there is no net charge $Q \equiv \int_V d^3x\,\rho_q$ from the combination of the test charge and its polarization cloud — see Problem 1.7. The difference between ϕ_t and the Coulomb potential is due to the collective *Debye shielding* effect.

We now use the result obtained in (1.10) to check that the expansion (1.4) was valid. Considering for simplicity a plasma with $T_i >> T_e$ [so the electron Debye length dominates in (1.8)], the ratio of the potential around an electron test charge to the electron temperature at the mean electron separation distance of $|\mathbf{x} - \mathbf{x}_t| = n_e^{-1/3}$ can be written as

$$\left.\frac{e\phi_t}{T_e}\right|_{|\mathbf{x}-\mathbf{x}_t|\,=\,n_e^{-1/3}} = \frac{\exp\left[-1/\left(n_e\lambda_{De}^3\right)^{1/3}\right]}{4\pi\,(n_e\lambda_{De}^3)^{2/3}} \simeq \frac{1}{4\pi\,(n_e\lambda_{De}^3)^{2/3}}. \qquad (1.11)$$

For this to be small and validate our expansion in (1.4), we must require

$$\boxed{n_e\lambda_D^3 >> 1, \quad \text{necessary condition for the plasma state.}} \tag{1.12}$$

That is, we must have many charged particles (electrons) within a *Debye cube* — a cube each side of which is the Debye shielding distance in length.[5] Physically, (1.12) is a necessary condition for the plasma state because it represents the requirement that, at the mean interparticle separation distance, collective interactions of charged particles dominate over binary interactions. The number of charged particles within a Debye cube (or more often its reciprocal $1/n_e\lambda_D^3$) is called the *plasma parameter* since it must be large for the medium to be in the plasma state.

As another check on the validity of the preceding expansion approach, we next confirm that the electric field energy in the polarization cloud is small compared to a typical thermal energy for the test particle — the temperature of that species of particles. The polarization electric field is determined by the difference between the potential ϕ_t around a test charge in the plasma and the test charge's Coulomb potential $\phi_{\rm Coul}$:

$$\mathbf{E}_{\rm pol} = -\boldsymbol{\nabla}\left(\phi_t - \phi_{\rm Coul}\right) = -\,\hat{\mathbf{e}}_{\rm r}\frac{d}{dr}\left[\frac{q\left(e^{-r/\lambda_D}-1\right)}{\{4\pi\epsilon_0\}r}\right] \tag{1.13}$$

in which $r \equiv |\mathbf{x}-\mathbf{x}_t|$ and $\hat{\mathbf{e}}_{\rm r} \equiv \boldsymbol{\nabla} r = (\mathbf{x}-\mathbf{x}_t)/|\mathbf{x}-\mathbf{x}_t|$ is a unit vector in the $\mathbf{r} \equiv \mathbf{x}-\mathbf{x}_t$ direction. The variation of the polarization electric field as a function of the distance r away from the test charge is shown in Fig. 1.2.

The energy density associated with this electric field is $\epsilon_0|\mathbf{E}_{\rm pol}|^2/2$. Using a spherical coordinate system whose origin is at the position of the test charge, we find that the total electric field energy obtained by integrating the energy density, normalized to the electron temperature (again assuming $T_i >> T_e$ for simplicity) can be written as

$$\begin{aligned}\frac{1}{T_e}\int d^3x\,\frac{\epsilon_0}{2}|\mathbf{E}_{\rm pol}|^2 &= \frac{4\pi\epsilon_0}{2T_e}\left(\frac{q}{\{4\pi\epsilon_0\}}\right)^2\int_0^\infty r^2dr\left[\frac{d}{dr}\left(\frac{e^{-r/\lambda_D}-1}{r}\right)\right]^2\\ &\equiv \frac{I}{8\pi n_e\lambda_D^3}.\end{aligned} \tag{1.14}$$

Here, the dimensionless integral I is simplified using $x \equiv r/\lambda_D$ and is given by

$$\begin{aligned} I &\equiv \int_0^\infty dx\left[x\frac{d}{dx}\left(\frac{e^{-x}-1}{x}\right)\right]^2 = \int_0^\infty dx\left[e^{-x}-\frac{1-e^{-x}}{x}\right]^2\\ &= \int_0^\infty dx\left[e^{-2x}-\frac{2}{x}\left(e^{-x}-e^{-2x}\right)+\frac{\left(1-e^{-x}\right)^2}{x^2}\right] = \int_0^\infty dx\,e^{-2x} = \frac{1}{2}\end{aligned}$$

[5]Since the intrinsic geometry of the polarization cloud around a test charge is spherical, plasma physicists often use as the appropriate measure the number of charged particles within a Debye sphere, $(4\pi/3)n_e\lambda_D^3$, which by (1.12) must also be large compared to unity.

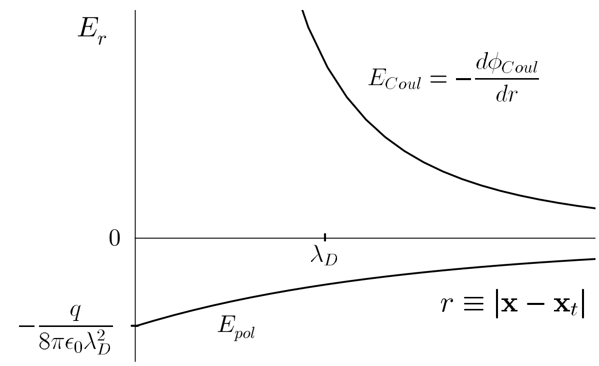

Figure 1.2: Coulomb and polarization radial electric fields around a test particle of charge q in a plasma. Because the plasma polarization acts to shield out the positive Coulomb electric field, the polarization electric field is negative. The polarization electric field is finite at the origin, decays smoothly with distance away from the test charge, and shields out the Coulomb electric field for separations larger than the Debye length λ_D.

in which in the first integral form on the second line we have integrated the last term by parts and cancelled it with the second term, and the final integral is evaluated using (**??**) in Appendix C. From (1.14) we again see that our expansion approach is valid as long as there are many electrons within a Debye cube (or sphere), since then the electric field energy in the polarization cloud around a test charge is small compared to the typical, thermal energy of a charged particle in a plasma.

We can also use the concepts developed in the preceding discussion to estimate the level of *thermal fluctuations* or noise in a plasma. The thermal fluctuations are caused by the interactions between charged particles through the electric field around one particle influencing the positions of other particles within approximately a Debye sphere around the original charged particle. That is, they are caused by correlations between particles, or by electric field correlations within the plasma. To calculate these properly requires a plasma kinetic theory (see Chapter 13). However, the fluctuation level can be estimated as follows.

A relevant measure of the magnitude of the thermal noise in a plasma is the ratio of the electric field energy density in the fluctuations $\epsilon_0|\tilde{\mathbf{E}}|^2/2$ to the thermal energy density nT. The polarization electric field given by (1.13) represents the correlation electric field between a test particle at $\mathbf{x}_t$ and an observation point $\mathbf{x}$. From Fig. 1.2 we see that the polarization electric field is localized to within about a Debye length of any given charge, and its magnitude there is not too different from its value at $r \equiv |\mathbf{x} - \mathbf{x}_t| = 0$: $\mathbf{E}_{\rm pol}(0) = -q/(2\{4\pi\epsilon_0\}\lambda_D^2)$. Also, we note that all charged particles within about a Debye sphere [namely

$\sim (4\pi/3) n_e \lambda_D^3$ particles] will contribute to the electric field fluctuations at any given point. Hence, omitting numerical factors, we deduce that the scaling of the relative electric field fluctuation energy from "two-particle" correlations in a plasma is given by

$$\frac{\frac{\epsilon_0}{2}\,|\tilde{\mathbf{E}}|^2}{n_e T_e} \sim \frac{\left(\frac{4\pi}{3} n_e \lambda_D^3\right)\left[\frac{\epsilon_0}{2}\,|\mathbf{E}_{\rm pol}(0)|^2\right]}{n_e T_e} \sim \frac{1}{n_e \lambda_D^3} << 1, \quad \text{thermal fluctuation level.} \tag{1.15}$$

We thus see that, as long as (1.12) is satisfied, the thermal fluctuation level is small compared to the thermal energy density in the plasma and again our basic expansion approach is valid. The thermal fluctuations occur predominantly at scale-lengths of order the Debye length or smaller. The appropriate numerical factor to be used in this formula, and the frequency and wave-number dependence of the thermal fluctuations in a plasma, can be obtained from plasma kinetic theory. They will be discussed and determined in Chapter 13.

1.2 Boundary Conditions; Plasma Sheath

A plasma should be larger than the Debye shielding distance in order not to be dominated by boundary effects. However, at the edge of a plasma where it comes into contact with a solid material (e.g., a wall, the earth), boundary effects become important. The region where the transformation from the plasma state to the solid state takes place is called the *plasma sheath.*

The role of a plasma sheath can be understood as follows. First, note that for comparable electron and ion temperatures the typical electron speed, which will be taken to be the electron thermal speed $v_{Te} \equiv \sqrt{2T_e/m_e}$ [see (**??**) in Section A.3], is much larger than the typical (thermal) ion speed ($v_{Te}/v_{Ti} \sim \sqrt{m_i/m_e} \gtrsim 43 >> 1$). Since the electrons typically move much faster than the ions,[6] electrons tend to leave a plasma much more rapidly than ions. This causes the plasma to become positively charged and build up an equilibrium electrostatic potential that is large enough [$\sim$ a few T_e/e, see (1.23) below] to reduce the electron loss rate to the ion loss rate — so the plasma can be quasineutral in steady state. The potential variation is mostly localized to the plasma sheath region, which is of order a few Debye lengths in width because that is the scale length on which significant departures from charge neutrality are allowed in a plasma. Thus, a plasma in contact with a grounded wall will: charge up positively, and be quasineutral throughout most of the plasma, but have a positively charged plasma sheath region near the wall.

We now make these concepts more concrete and quantitative by estimating the properties of a one-dimensional sheath next to a grounded wall using a simple plasma model. Figure 1.3 shows the specific geometry to be considered along

[6] Many people use an analogy to remember that electrons have much larger thermal velocities than ions: electrons are like fast moving, lightweight ping pong balls while ions are like slow-moving, more massive billiard balls — for equal excitation or thermal energies.

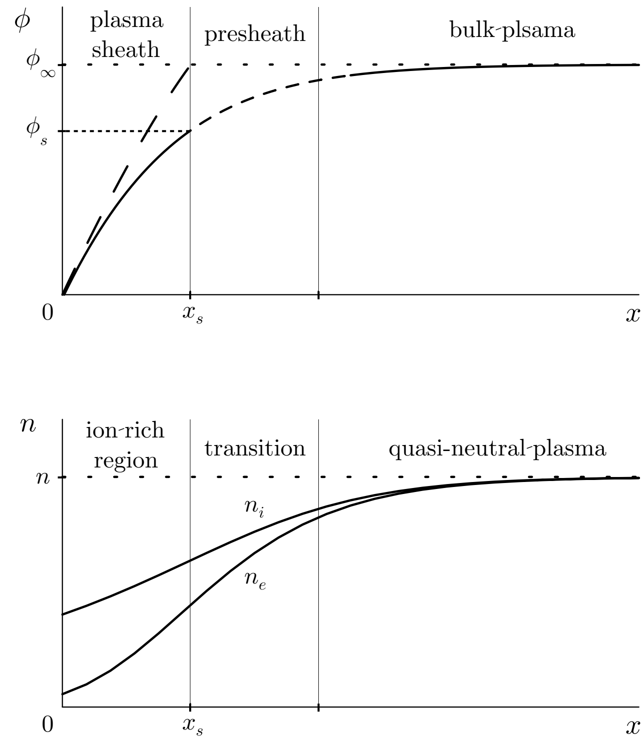

Figure 1.3: Behavior of the electrostatic potential and electron and ion densities in the sheath, presheath (or transition) and bulk plasma regions of a plasma in contact with a grounded wall. For the case shown $T_e/m_i V_\infty^2 = 0.9$. The sheath parameters determined in the text are $\Phi_\infty \simeq 3\,T_e/e$ and $x_S \simeq 2\,\lambda_{De}$. The long-dash line in the top figure indicates the approximation given in (1.26).

with the behavior of the potential, and electron and ion densities in the plasma sheath and bulk plasma regions, as well as in the presheath (or transition) region between them.

The electron density is determined from the Boltzmann relation (1.3):

$$n_e(x) = n_\infty \exp\left\{\frac{e\,[\Phi(x) - \Phi_\infty]}{T_e}\right\} \tag{1.16}$$

in which $\Phi(x)$ indicates the equilibrium potential profile in the plasma, and the ∞ subscript indicates evaluation of the quantities in the bulk plasma region far from the wall (i.e., beyond the plasma sheath and presheath regions whose properties we will determine). (In using this equation it is implicitly assumed that the background electron velocity distribution is Maxwellian.)

For simplicity, we consider an electron-proton plasma with negligible ion thermal motion effects ($e\Phi >> T_i$). The potential variation in and near the sheath produces an electric field that increases the flow of ions toward the wall, which will be assumed to be grounded. The ion flow speed V_i in the x direction is governed by conservation of energy for the "cold" ($e\Phi >> T_i$) ions:

$$\frac{1}{2}m_iV_i^2(x) + e\Phi(x) = \text{constant} = \frac{1}{2}m_iV_\infty^2 + e\Phi_\infty \tag{1.17}$$

in which we have allowed for a flow of ions from the bulk of the plasma into the presheath region so as to ultimately balance the electron flow to the wall. The ion flow at any given point is given by

$$V_i(x) = \sqrt{V_\infty^2 + \frac{2e}{m_i}\left[\Phi_\infty - \Phi(x)\right]} = \sqrt{\frac{2e}{m_i}\left[\Phi_\infty + \frac{m_iV_\infty^2}{2e} - \Phi(x)\right]}.$$

The spatial change in the ion flow speed causes the ion density to change as well — a high flow speed produces a low ion density. The ion density variation is governed, in a steady equilibrium, by the continuity or density conservation equation [see (**??**) in Appendix A] for the ion density: $d(n_iV_i)/dx = 0$, or $n_i(x)V_i(x) = \text{constant}$. Thus, referencing the ion density to its value n_∞ in the bulk plasma ($x \to \infty$), it can be written as

$$n_i(x) = n_\infty\left\{1 + \frac{2e\left[\Phi_\infty - \Phi(x)\right]}{m_iV_\infty^2}\right\}^{-1/2}. \tag{1.18}$$

Substituting the electron and ion densities into Poisson's equation (1.1), we obtain the equation that governs the spatial variation of the potential in the sheath, presheath and plasma regions:

$$\begin{aligned}\frac{d^2\Phi}{dx^2} &= -\frac{e}{\epsilon_0}(n_i - n_e)\\ &= -\frac{n_\infty e}{\epsilon_0}\left[\left\{1 + \frac{2e\left[\Phi_\infty - \Phi(x)\right]}{m_iV_\infty^2}\right\}^{-1/2} - \exp\left\{-\frac{e\left[\Phi_\infty - \Phi(x)\right]}{T_e}\right\}\right].\end{aligned} \tag{1.19}$$

While numerical solutions of this equation can be obtained, no analytic solution is available. However, limiting forms of the solution can be obtained near the wall ($x << x_S$) and in the bulk plasma ($x >> x_S$). Even though a simple solution is not available in the transition region, solutions outside this region can be used to define the sheath position x_S and the conditions needed for proper sheath formation.

In the quasineutral plasma far from the plasma sheath region ($x >> x_S$) the potential $\phi(x)$ is very close to its asymptotic value Φ_∞. In this region we approximate the electron and ion densities in the limits $2e\left[\Phi_\infty - \Phi(x)\right]/m_iV_\infty^2 << 1$ and $e\left[\Phi_\infty - \Phi(x)\right]/T_e << 1$, respectively:

$$\begin{aligned}n_e(x) &\simeq n_\infty\left\{1 - \frac{e\left[\Phi_\infty - \Phi(x)\right]}{T_e} + \cdots\right\},\\ n_i(x) &\simeq n_\infty\left\{1 - \frac{e\left[\Phi_\infty - \Phi(x)\right]}{m_iV_\infty^2} + \cdots\right\}.\end{aligned}$$

Keeping only linear terms in $\Phi_\infty - \Phi(x)$, (1.19) can thus be simplified to

$$\frac{d^2\left[\Phi_\infty - \Phi(x)\right]}{dx^2} \simeq \frac{1}{\lambda_{De}^2}\left(1 - \frac{T_e}{m_i V_\infty^2}\right)\left[\Phi_\infty - \Phi(x)\right]. \tag{1.20}$$

Here, λ_{De} is the electron Debye length evaluated at the bulk plasma density n_∞. For $m_i V_\infty^2 < T_e$ the coefficient of $\Phi_\infty - \Phi(x)$ on the right would be negative; this would imply a spatially oscillatory solution that is not physically realistic for the present plasma model, which implicitly assumes that the potential is a monotonic function of x. Thus, a necessary condition for proper sheath formation in this model is

$$\boxed{|V_\infty| \geq \sqrt{T_e/m_i}\,, \quad \text{Bohm sheath criterion.}} \tag{1.21}$$

Since this condition need only be satisfied marginally and the ion flow into the sheath region typically assumes its minimum value, it is usually sufficient to make this criterion an equality. The Bohm sheath criterion implies that ions must enter the sheath region sufficiently rapidly to compensate for the electron charge leakage through the sheath to the wall. In general, what is required for proper sheath formation is that, as we move toward the wall, the local charge density increases as the potential decreases: $\partial\rho_q/\partial\Phi < 0$ for all x. Also, since we will later find (see Section 1.4) that $\sqrt{T_e/m_i}$ is the speed of ion acoustic waves in a plasma (for the plasma model being considered), the Bohm sheath criterion implies that the ions must enter the presheath region at a supersonic speed relative to the ion acoustic speed.

As long as the Bohm sheath criterion is satisfied, solutions of (1.19) will be well-behaved, and exponentially damped in the presheath region: for $x >> x_S$ we have $\Phi_\infty - \Phi(x) \simeq C\exp(-x/h)$ where $h = \lambda_{De}(1 - T_e/m_i V_\infty^2)^{-1/2}$ and C is a constant of order $\Phi_\infty - \Phi_S$. Thus, for this plasma model, in the typical case where V_∞ is equal to or slightly exceeds $\sqrt{T_e/m_i}$, the presheath region where the potential deviates from Φ_∞ extends only a few Debye lengths into the plasma. In more comprehensive models for the plasma, and in particular when ion thermal effects are included, it is found that the presheath region can be larger and the potential variation in this region is influenced by the effects of sheath geometry, local plasma sources, collisions and a magnetic field (if present). However, the Bohm sheath criterion given by (1.21) remains unchanged for most physically relevant situations, as long as the quantity on the right side is interpreted to be the ion acoustic speed in the plasma model being used.

We next calculate the *plasma potential* Φ_∞ that the plasma will rise to in order to hold back the electrons so that their loss rate will be equal to the ion loss rate from the plasma. The flux of ions to the wall is given by $-n_i V_\infty$, which when evaluated at the Bohm sheath criterion value given in (1.21) becomes $-n_\infty\sqrt{T_e/m_i}$. (The flux is negative because it is in the negative x direction.) A Maxwellian distribution of electrons will produce (see Section A.3) a random flux of electrons to the wall on the left side of the plasma of $-(1/4)n_e\bar{v}_e = -(n_\infty/4)\exp(-e\Phi_\infty/T_e)\sqrt{8T_e/\pi m_e}$. Thus, the net electrical current density to the wall will be given by

$$\begin{aligned} J &= J_i - J_e = -e(n_i V_\infty - n_e \bar{v}_e) \\ &= -en_\infty \left[\sqrt{T_e/m_i} - (1/4)\sqrt{8T_e/\pi m_e} \exp\left(-e\Phi_\infty/T_e\right) \right]. \end{aligned} \tag{1.22}$$

Since in a quasineutral plasma equilibrium we must have $J = 0$, the plasma potential Φ_∞ is given in this plasma model by

$$\boxed{\Phi_\infty = \frac{T_e}{e} \ln \sqrt{\frac{m_i}{2\pi m_e}} \geq 2.84\,\frac{T_e}{e} \simeq 3\,\frac{T_e}{e}, \quad \text{plasma potential,}} \tag{1.23}$$

where after the inequality we have used the proton to electron mass ratio m_i/m_e = 1836. In the original work in this area in 1949, Bohm argued that a potential drop of $T_e/2e$ extending over a long distance into the plasma (much further than where we are calculating) is required to produce the incoming ion speed $V_\infty \geq \sqrt{T_e/m_i}$ at the sheath edge. In Bohm's model the density n_∞ is $e^{-1/2} =$ 0.61 times smaller than the bulk plasma ion density and thus the ion current J_i is smaller by this same factor. For this model, the potential Φ_∞ in (1.23) increases by $0.5\,T_e/e$ to $3.34\,T_e/e$. Since lots additional physics (see end of preceding paragraph) needs to be included to precisely determine the plasma potential for a particular situation, and the plasma potential does not change too much with these effects, for simplicity we will take the plasma potential Φ_∞ to be approximately $3\,T_e/e$.

Finally, we investigate the form of $\Phi(x)$ in the sheath region near the wall ($x << x_S$). In this region the potential is much less than Φ_∞ and the electron density becomes so small relative to the ion density that it can be neglected. The equation governing the potential in this ion-rich region can thus be simplified from (1.19) to

$$\frac{d^2\Phi(x)}{dx^2} \simeq -\frac{en_\infty}{\epsilon_0}\left[\frac{m_i V_\infty^2/2e}{\Phi_\infty + m_i V_\infty^2/2e - \Phi(x)}\right]^{1/2}. \tag{1.24}$$

This equation can be de-dimensionalized by multiplying through by e/T_e. Thus, defining a dimensionless potential variable χ by

$$\chi(x) \equiv \frac{e\left[\Phi_\infty + m_i V_\infty^2/2e - \Phi(x)\right]}{T_e}, \tag{1.25}$$

the equation can be written as

$$\frac{d^2\chi}{dx^2} \simeq \frac{1}{\delta^2\sqrt{\chi}}.$$

in which $\delta \equiv \lambda_{De}/(m_i V_\infty^2/2T_e)^{1/4}$.

To integrate this equation we multiply by $d\chi/dx$ and integrate over x using $(d\chi/dx)(d^2\chi/dx^2) = (1/2)(d/dx)(d\chi/dx)^2$ and $dx(d\chi/dx)/\sqrt{\chi} = d\chi/\sqrt{\chi} = 2d\sqrt{\chi}$ to obtain

$$\frac{1}{2}\left(\frac{d\chi}{dx}\right)^2 \simeq \frac{2}{\delta^2}\sqrt{\chi} + \text{constant.}$$

Since both χ and $d\chi/dx$ are small near x_S, this equation is approximately valid for the $x < x_S$ region if we take the constant in it to be zero. Solving the resultant equation for $d\chi/dx$, we obtain

$$\frac{d\chi}{dx} \simeq -\frac{2\chi^{1/4}}{\delta} \quad \Longrightarrow \quad \frac{4}{3}d(\chi^{3/4}) \simeq -\frac{2dx}{\delta}.$$

Integrating this equation from $x = 0$ where $\chi = \chi_0 \equiv (e\Phi_\infty + m_i V_\infty^2/2)/T_e$ to x where $\chi = \chi(x)$, we obtain

$$\chi^{3/4}(x) - \chi_0^{3/4} \simeq -\frac{3x}{2\delta},$$

or

$$\Phi(x) \simeq (\Phi_\infty + m_i V_\infty^2/2e)[1 - (1 - x/x_S)^{4/3}]. \tag{1.26}$$

Here, we have defined

$$x_S \equiv \frac{2\delta\chi_0^{3/4}}{3} = \frac{2^{5/4}}{3}\left(\frac{T_e}{m_i V_\infty^2}\right)^{1/4}\left(\frac{e\Phi_\infty + m_i V_\infty^2/2}{T_e}\right)^{3/4}\lambda_{De}, \quad \text{sheath thickness.} \tag{1.27}$$

Equation (1.26) is valid in the sheath region near the wall ($0 < x << x_S$). We have identified the scale length in (1.27) with the sheath width x_S because this is the distance from the wall at which the potential $\Phi(x)$ extrapolates to the effective plasma potential in the bulk plasma, $\Phi_\infty + m_i V_\infty^2/2e$.

Using the value for Φ_∞ given in (1.23) and $V_\infty \simeq \sqrt{T_e/m_i}$, the sheath thickness becomes $x_S \simeq 2\,\lambda_{De}$. Thus, as shown in Fig. 1.3, for this model the plasma charges to a positive potential of a few T_e/e and is quasineutral up to the non-neutral plasma sheath region, which extends a few Debye lengths ($\sim 2\,x_S \sim 4\,\lambda_{De}$ in Fig. 1.3) from the grounded wall into the plasma region.

1.3 Langmuir Probe Characteristics*

To further illuminate the electrical properties of a static or equilibrium plasma, we next determine the current that will be drawn out of a probe inserted into an infinite plasma and biased to a voltage or potential Φ_B. Such probes provided some of the earliest means of diagnosing plasmas and are called *Langmuir probes*, after Irving Langmuir who developed much of the original understanding of their operation. The specific situation to be considered is sketched in Fig. 1.4. For simplicity we assume that the probe is small compared to the size of the plasma and does not significantly disturb it. The probe will be assumed to have a metallic (e.g., molybdenum) tip and be electrically connected to the outside world via an insulated tube through the plasma. Probes of this type are

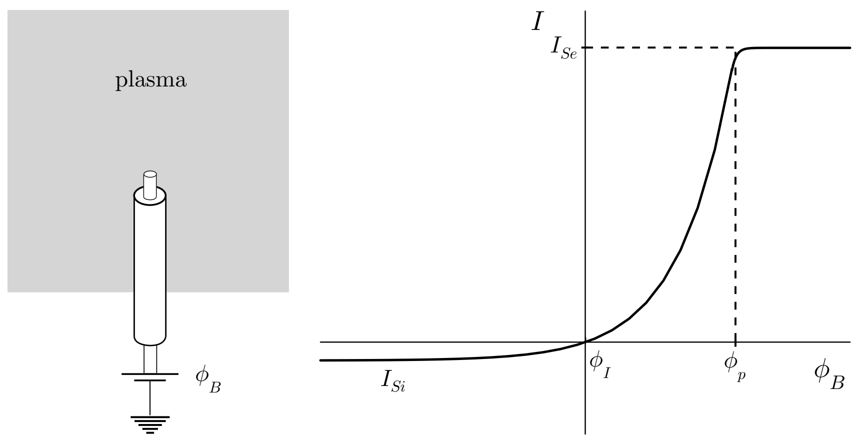

Figure 1.4: Schematic of Langmuir probe inserted into a plasma and its idealized current-voltage characteristics: current I drawn out of the probe as a function of the bias voltage or potential Φ_B. The labeled potentials and currents are: Φ_f, floating potential; Φ_p, plasma potential; I_{Si}, ion saturation current; I_{Se}, electron saturation current.

often used in laboratory plasmas that have modest parameters ($T_e \lesssim 10$ eV, $n_e \lesssim 10^{19}$ m^{-3} — probes tend to get burned up at higher plasma parameters).

Since the bias potential Φ_B on the probe will not affect the incoming ion flow speed V_∞ (for $\Phi_B < \Phi_\infty$), following the discussion leading to (1.22) the ion current out of the probe will be given by

$$I_i = A_S J_i = -n_\infty e\sqrt{T_e/m_i}\, A_S \equiv -I_{Si}, \quad \text{ion saturation current } (I_{Si}), \tag{1.28}$$

where A_S is the area of the probe plus sheath over which the ions are collected by the probe. For the electrons we must take account of the bias potential Φ_B on the probe. The electron current into the probe is given by

$$I_e = A_p J_e = \begin{cases} n_\infty e\sqrt{T_e/2\pi m_e}\, A_p \equiv I_{Se}, & \Phi_B \geq \Phi_p, \\ n_\infty e\sqrt{T_e/2\pi m_e}\, A_p \exp\left[-e(\Phi_p - \Phi_B)/T_e\right], & \Phi_B < \Phi_p, \end{cases} \tag{1.29}$$

in which A_p is the area of the probe and Φ_p is the plasma potential — the voltage at which all electrons heading toward the probe are collected by it. (Whereas the effective area for ions to be collected by the probe encompasses both the probe and the sheath, for $\Phi_B < \Phi_p$ the relevant area A_p for electrons is just the probe area since only those electrons surmounting the sheath potential make it to the probe — see Fig. 1.3. However, when $\Phi_B > \Phi_p$ the relevant area, and consequent electron current, grows slightly and roughly linearly with bias voltage, which is then attracting electrons and modifying their trajectories in the vicinity of the probe. In the idealized current-voltage curve in Fig. 1.4 we have neglected this latter effect.)

The total current $I = I_i + I_e$ drawn to the probe is shown in Fig. 1.4 as a function of the bias voltage or potential Φ_B. For a large negative bias the electron current becomes negligible and the current is totally given by the ion current I_{Si}, which is called the *ion saturation current.* The potential Φ_f at which the current from the probe vanishes is called the *floating potential*, which is zero for our simple model. However, it is often slightly negative in real plasmas, unless there is secondary electron emission from the probe, in which case it can become positive. For potentials larger than the *plasma potential* Φ_p all electrons on trajectories that intercept the probe are collected and the current is given by the *electron saturation current* I_{Se}. Except for differences in the charged particle collection geometry (typically cylindrical or spherical probes versus a planar wall), in the sheath thickness (relative to probe size) effects and perhaps in secondary electron emission, the difference between the plasma and floating potentials is just the naturally positive plasma potential that we derived in (1.23). That is, $\Phi_p - \Phi_f \simeq \Phi_\infty \sim 3\,T_e/e$.

In a real plasma the idealized current-voltage characteristic that is indicated in Fig. 1.4 gets rounded off and distorted somewhat due to effects such as charged particle orbit effects in the sheath, probe geometry, secondary electron emission from the probe and other effects. Indeed, because of the practical importance of Langmuir probes in measuring plasma parameters in many laboratory plasmas, as we will discuss in the next paragraph, there is a large literature on the current-voltage characteristics of various types of probes in real plasmas (see references and suggested reading at the end of the chapter). Nonetheless, the basic characteristics are as indicated in Fig. 1.4.

For bias potentials that lie between the floating and plasma potentials, the current from the probe increases exponentially with bias potential Φ_B. Thus, the electron temperature can be deduced from the rate of exponential growth in the current as the bias potential is increased: $T_e/e \simeq (I - I_{Si})/(dI/d\Phi_B)$. Alternatively, one can use a "double probe" to determine the electron temperature — see Problem 1.11. If the electron temperature is known, the plasma ion density can be estimated from the ion saturation current: $n_\infty \simeq I_{Si}/(eA_S\sqrt{T_e/M_i})$. Langmuir probes are thus important diagnostics for measuring the plasma density and electron temperature in laboratory plasmas with modest parameters.

The thickness of the plasma sheath changes as the bias potential Φ_B is varied. The derivation of the sheath thickness x_S given in (1.24) to (1.27) can be modified to account for the present biased probe situation by replacing the potential Φ_∞ with $\Phi_p - \Phi_B$. Thus, setting $m_i V_\infty^2/T_e$ to unity to satisfy the Bohm sheath criterion (1.21), the sheath thickness around a biased probe in a plasma is given approximately by

$$x_S \simeq \frac{2^{5/4}}{3}\left(\frac{\Phi_p + 0.5 - \Phi_B}{T_e/e}\right)^{3/4} \lambda_{De}, \quad \text{sheath thickness.} \tag{1.30}$$

This formula is valid for $e(\Phi_p + 0.5 - \Phi_B) >> T_e$ — small or negative bias voltages $\Phi_B - \Phi_p$. As the bias potential Φ_B increases toward the plasma potential Φ_p, the plasma sheath becomes thinner; it disappears for $e(\Phi_p - \Phi_B) \lesssim 0.5\, T_e$.

For large negative bias potentials ($|\Phi_B| >> T_e/e$), the electrical current density flowing through the ion-rich sheath region is limited by "space charge" effects and given by the Child-Langmuir law — see Problem 1.13. However, transiently the current density can be larger that indicated by the Child-Langmuir law — see Problem 1.14.

1.4 Inertial Response; Plasma Oscillations

In the preceding sections on Debye shielding and its effects we considered the adiabatic or static response of charged particles and a plasma to the Coulomb electric field around a charged particle in the plasma. Next, we discuss the *inertial* (or dynamic) response of a plasma. To do this we consider the electric polarization response of charged particles and a plasma to a small electric field perturbation, which may be externally imposed or be collectively generated within the plasma.

First, we calculate the motion of a charged particle in response to an electric field. The velocity $\mathbf{v}$ of a charged particle of mass m and charge q subjected to an electric field perturbation[7] $\tilde{\mathbf{E}}(\mathbf{x}, t)$ is governed by Newton's second law ($\mathbf{F} = m\mathbf{a}$) with force $q\tilde{\mathbf{E}}$:

$$m\,\frac{d\mathbf{v}}{dt} = q\,\tilde{\mathbf{E}}(\mathbf{x}, t). \tag{1.31}$$

The electric field perturbation will be assumed to be small enough and sufficiently slowly varying in space so that nonlinear and translational motion effects are negligible. Thus, it will be sufficient to evaluate the electric field at the initial position $\mathbf{x}_0$ and neglect the small variation in the electric field induced by the motion $\mathbf{x}(t)$ of the charged particle. This approximation will be discussed further after the next paragraph.

Integrating (1.31) over time, the velocity perturbation $\tilde{\mathbf{v}}(t)$ induced by the electric field perturbation for a particle with initial velocity $\mathbf{v}_0$ is given by

$$\tilde{\mathbf{v}}(t) \equiv \mathbf{v}(t) - \mathbf{v}_0 = \frac{q}{m}\int_0^t dt'\;\tilde{\mathbf{E}}[\mathbf{x}'(t'), t'] \simeq \frac{q}{m}\int_0^t dt'\;\tilde{\mathbf{E}}(\mathbf{x}_0, t'). \tag{1.32}$$

Integrating once more over time, we find that the motion induced by the electric field perturbation becomes

$$\tilde{\mathbf{x}}(t) \equiv \mathbf{x}(t) - (\mathbf{x}_0 + \mathbf{v}_0 t) \simeq \frac{q}{m}\int_0^t dt' \int_0^{t'} dt''\;\tilde{\mathbf{E}}(\mathbf{x}_0, t''), \quad \text{inertial response.} \tag{1.33}$$

[7] Perturbations to an equilibrium will be indicated throughout the book by a tilde over the symbol for the perturbed quantity. Equilibrium quantities will be indicated by 0 (zero) subscripts.

Because the response of the particle to the electric field force is limited by the inertial force $m\,\mathbf{a} = m\,d\mathbf{v}/dt$, this is called an *inertial response.* Note that this response is inversely proportional to the mass of the charged particle; thus, the lighter electrons will give the primary inertial response to an electric field perturbation in a plasma.

We check our approximation of evaluating the electric field at the initial position $\mathbf{x}_0$ by expanding the electric field in a Taylor series expansion about the charged particle trajectory given by

$$\tilde{\mathbf{E}}\,[\mathbf{x}(t), t] = \tilde{\mathbf{E}}\,(\mathbf{x}_0, t) + (\tilde{\mathbf{x}} + \mathbf{v}_0 t) \cdot \boldsymbol{\nabla}\tilde{\mathbf{E}}\,|_{\mathbf{x}_0} + \cdots \tag{1.34}$$

Our approximation is valid as long as the second (and higher order) terms in this expansion are small compared to the first term:

$$(\tilde{\mathbf{x}} + \mathbf{v}_0 t) \cdot \boldsymbol{\nabla}\tilde{\mathbf{E}} << \tilde{\mathbf{E}}. \tag{1.35}$$

Thus, the electric field perturbation must vary sufficiently slowly in space (i.e., the gradient scale length $|(1/|\tilde{\mathbf{E}}|)\boldsymbol{\nabla}\tilde{\mathbf{E}}|^{-1}$ must be long compared to the distance $|\tilde{\mathbf{x}} + \mathbf{v}_0 t|$), be small enough (so the nonlinear term $\tilde{\mathbf{x}} \cdot \boldsymbol{\nabla}\tilde{\mathbf{E}}$ is small compared to $\tilde{\mathbf{E}}$) and the elapsed time must not be too long. These approximations will be checked *a posteriori* — at the end of this section.

The inertial motion $\tilde{\mathbf{x}}$ of a charged particle in response to the electric field perturbation creates an electric dipole moment $q\tilde{\mathbf{x}}$. A uniform density n_0 of such charged particles leads to an electric polarization density $\tilde{\mathbf{P}} = n_0 q\tilde{\mathbf{x}}$. Summing over the species of charged particles in the plasma, the total plasma polarization density becomes

$$\tilde{\mathbf{P}} = \sum_s n_{0s} q_s \tilde{\mathbf{x}}_s = \epsilon_0 \sum_s \omega_{ps}^2 \int_0^t dt' \int_0^{t'} dt''\,\tilde{\mathbf{E}}\,(t'') \tag{1.36}$$

in which for each charged species s

$$\omega_{ps}^2 \equiv \frac{n_{0s} q_s^2}{m_s \epsilon_0}, \quad \text{square of species plasma frequency,} \tag{1.37}$$

is the inertial or *plasma frequency* for a species s, whose physical significance will be discussed below.

Because the ions are so much more massive than the electrons (the ratio of the proton to electron mass is 1836), they have much more inertia. Thus, their plasma frequency is much smaller than that for the electrons — for example, for protons $\omega_{pi}/\omega_{pe} = \sqrt{m_e/m_p} \simeq 1/43 << 1$. Since the electrons give the dominant contribution to the plasma polarization and have the largest plasma frequency, we have

$$\sum_s \omega_{ps}^2 = \omega_{pe}^2 + \omega_{pi}^2 \simeq \omega_{pe}^2. \tag{1.38}$$

Numerically, the electron plasma frequency is given by

$$\boxed{\omega_{pe} \equiv \sqrt{\frac{n_e e^2}{m_e \epsilon_0}} \simeq 56\sqrt{n_e(\mathrm{m}^{-3})}\ \text{rad/sec, radian plasma frequency,}} \qquad (1.39)$$

or

$$f_{pe} \equiv \omega_{pe}/2\pi \simeq 9\sqrt{n_e(\mathrm{m}^{-3})}\ \mathrm{Hz}, \quad \text{plasma frequency.} \qquad (1.40)$$

The plasma polarization in (1.36) causes [see(**??**) and (**??**) Section A.2] a polarization charge density $\tilde{\rho}_{\mathrm{pol}}$ given by the negative of the divergence of the polarization $\tilde{\mathbf{P}}$:

$$\tilde{\rho}_{\mathrm{pol}}(\tilde{\mathbf{E}}) = -\boldsymbol{\nabla\cdot}\tilde{\mathbf{P}} = -\,\epsilon_0 \sum_s \omega_{ps}^2 \int_0^t dt' \int_0^{t'} dt''\ \boldsymbol{\nabla\cdot}\tilde{\mathbf{E}}(t''). \qquad (1.41)$$

Now, to calculate the perturbed electric field $\tilde{\mathbf{E}}$ in a plasma we need to use Gauss's law, which is given in (1.1). For the charge density ρ_q we imagine that there are polarization charge densities due to both the electric field perturbation we have been considering, and an externally imposed electric field $\mathbf{E}_{\mathrm{ext}}$ which satisfies the same conditions as $\tilde{\mathbf{E}}$ — namely condition (1.35). Thus, the relevant form of Gauss's law becomes

$$\boldsymbol{\nabla\cdot}\tilde{\mathbf{E}} = \frac{1}{\epsilon_0}\tilde{\rho}_{\mathrm{pol}} = -\sum_s \omega_{ps}^2 \int_0^t dt' \int_0^{t'} dt''\ \boldsymbol{\nabla\cdot}\left[\tilde{\mathbf{E}}(t'') + \mathbf{E}_{\mathrm{ext}}(t'')\right]. \qquad (1.42)$$

This differential and integral equation in space and time, respectively, can be reduced to a simpler, completely differential form by taking its second partial derivative with respect to time to yield

$$\boldsymbol{\nabla\cdot}\left[\frac{\partial^2\tilde{\mathbf{E}}}{\partial t^2} + \sum_s \omega_{ps}^2\left(\tilde{\mathbf{E}} + \mathbf{E}_{\mathrm{ext}}\right)\right] = 0. \qquad (1.43)$$

Using the approximation in (1.38), we thus find that taking into account the inertial effects of charged particles (mostly electrons), nontrivial (i.e., nonvanishing) electric field perturbations satisfying condition (1.35) are governed by the differential equation

$$\frac{\partial^2\tilde{\mathbf{E}}}{\partial t^2} + \omega_{pe}^2\tilde{\mathbf{E}} = -\,\omega_{pe}^2\mathbf{E}_{\mathrm{ext}}. \qquad (1.44)$$

This is a linear, inhomogeneous differential equation of the harmonic oscillator type with frequency ω_{pe} for the perturbed electric field $\tilde{\mathbf{E}}$ induced by the externally applied electric field $\mathbf{E}_{ext}$.

The "complementary" (in the current langauge of mathematics) solutions of the homogeneous part of this equation are of the form

$$\tilde{\mathbf{E}}_h = \mathbf{C}_{\mathrm{c}}\cos\omega_{pe}t + \mathbf{C}_{\mathrm{s}}\sin\omega_{pe}t, \qquad (1.45)$$

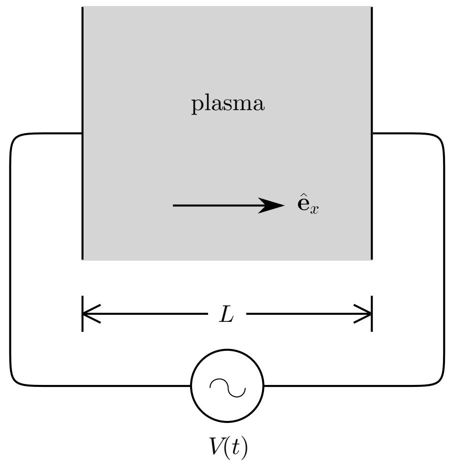

Figure 1.5: Schematic of circuit for imposing an oscillating potential $\Phi(t) = \Phi_0 \sin \omega_0 t$ across a plasma.

where $\mathbf{C}_\mathrm{c}$ and $\mathbf{C}_\mathrm{s}$ are arbitrary coefficient vectors to be fixed by the boundary conditions. These *plasma oscillation* solutions show that the plasma responds inertially to electric field perturbations by oscillating at the electron plasma frequency ω_{pe}. Externally imposed electric fields will induce perturbations in the plasma that are combinations of the time dependence of the externally imposed field and the electron plasma oscillations.

In the present simple model plasma oscillations are undamped. Collisions (electron-neutral or Coulomb) damp them at rates proportional to the relevant collision frequency ν — see Problem 1.18. Also, as we will discuss in Chapter 8, kinetic effects will lead to evanescence of these oscillations due to wave-particle resonance effects — Landau damping.

To illustrate the plasma responses more concretely, we consider the response of a plasma to an externally imposed sinusoidal electric field. (An alternative illustration for just plasma oscillations is developed in Problem 1.19 using a one-dimensional plasma slab model.) As shown in Fig. 1.5, the electric field will be induced by imposing an oscillating potential $\Phi(t) = \Phi_0 \sin \omega_0 t$ at time $t = 0$ across plates on opposite sides of a plasma of thickness L (implicitly $>> \lambda_D$) in the x direction. For simplicity the plasma will be assumed to be infinite in extent (or at least $>> L$) in the other two directions so that their effects can be neglected. Thus, the applied electric field will be given for $t > 0$ by

$$\mathbf{E}_\mathrm{ext} = \frac{\Phi_0}{L} \hat{\mathbf{e}}_\mathrm{x} \sin \omega_0 t \equiv \mathbf{E}_0 \sin \omega_0 t. \tag{1.46}$$

The particular solution of (1.44) in response to this externally applied electric field is

$$\tilde{\mathbf{E}}_p = \frac{\omega_{pe}^2}{\omega_0^2 - \omega_{pe}^2}\,\mathbf{E}_0 \sin\omega_0 t. \tag{1.47}$$

Adding together the homogeneous, particular and externally applied electric field components ($\mathbf{E} = \tilde{\mathbf{E}} + \mathbf{E}_{\rm ext} = \tilde{\mathbf{E}}_h + \tilde{\mathbf{E}}_p + \mathbf{E}_{\rm ext}$) of the solution of (1.44), and subjecting them to the boundary conditions that $\mathbf{E}(t=0) = \mathbf{0}$ and $d\mathbf{E}/dt\,|_{t=0} = d\mathbf{E}_{\rm ext}/dt\,|_{t=0} = \omega_0\mathbf{E}_0$, we find $\mathbf{C}_{\rm c} = \mathbf{0}$ and $\mathbf{C}_{\rm s} = -\left[\omega_0\omega_{pe}/\left(\omega_0^2 - \omega_{pe}^2\right)\right]\mathbf{E}_0$. Hence, the total electric field $\mathbf{E}$ driven by $\mathbf{E}_{ext}$ is given for $t > 0$ by

$$\begin{aligned}
\mathbf{E}(t) &= \frac{\omega_0\omega_{pe}}{\omega_0^2 - \omega_{pe}^2}\,\mathbf{E}_0 \sin\omega_{pe}t + \frac{\omega_{pe}^2}{\omega_0^2 - \omega_{pe}^2}\,\mathbf{E}_0 \sin\omega_0 t + \mathbf{E}_0 \sin\omega_0 t \\
&= -\frac{\omega_0\omega_{pe}}{\omega_0^2 - \omega_{pe}^2}\,\mathbf{E}_0 \sin\omega_{pe}t + \frac{\omega_0^2}{\omega_0^2 - \omega_{pe}^2}\,\mathbf{E}_0 \sin\omega_0 t \\
&\equiv \hat{\mathbf{E}}_{\rm plasma} \sin\omega_{pe}t + \hat{\mathbf{E}}_{\rm driven} \sin\omega_0 t.
\end{aligned} \tag{1.48}$$

The frequency dependences of the net driven response $\hat{\mathbf{E}}_{\rm driven}$ oscillating at frequency ω_0 and of the response $\hat{\mathbf{E}}_{\rm plasma}$ oscillating at the plasma frequency ω_{pe} are shown in Fig. 1.6. For ω_0 much less than the electron inertial or plasma frequency ω_{pe}, we find that $\hat{\mathbf{E}}_{\rm driven}$ is of order $-\omega_0^2/\omega_{pe}^2$ compared with the externally applied electric field $\mathbf{E}_0 \sin\omega_0 t$, and hence tends to be small. In this limit the electrons have little inertia ($\omega_0 << \omega_{pe}$) and they develop a strong polarization response that tends to collectively shield out the externally applied electric field from the bulk of the plasma. In the opposite limit $\omega_0^2 >> \omega_{pe}^2$, the inertia of the electrons prevents them from responding significantly, their polarization response is small, and the externally imposed electric field permeates the plasma — in this limit $\mathbf{E} \simeq \mathbf{E}_{\rm ext}$ since $\tilde{\mathbf{E}} << \mathbf{E}_{\rm ext}$. The singularity at $\omega_0 = \omega_{pe}$ indicates that when the driving frequency ω_0 coincides with the natural plasma oscillation frequency ω_{pe} the linear response is unbounded. In Chapters 7 and 8 we will see that collisions or kinetic effects bound this response and lead to weak damping effects for $\omega_0 \simeq \omega_{pe}$. Nonlinear effects can also lead to bounds on this response.

The $\hat{\mathbf{E}}_{\rm plasma}$ response in (1.48), which oscillates at the plasma frequency, is caused by the electron inertia effects during the initial turn-on of the external electric field. Note that it vanishes in both the low and high frequency limits — because for low ω_0 the excitation is small for the nearly adiabatic ($\omega_0 << \omega_{pe}$) turn-on phase, while for high ω_0 the electron inertial response is small during the very brief ($\delta t \sim 1/\omega_0 << 1/\omega_{pe}$) turn-on phase. Like the driven response, the plasma response becomes unbounded in this simple plasma model for $\omega_0 \to \omega_{pe}$.

Finally, we go back and determine the conditions under which the approximation (1.35) that we made in calculating the plasma polarization induced by an electric field is valid. Referring to the physical situation shown in Fig. 1.5, we take the gradient scale length of the electric field perturbation $|(1/|\tilde{\mathbf{E}}|)\boldsymbol{\nabla}\tilde{\mathbf{E}}|^{-1}$ to be of order the spacing L between the plates. We first estimate the condition imposed by the particle streaming indicated by the term $\mathbf{v}_0 t$ in (1.35). To make

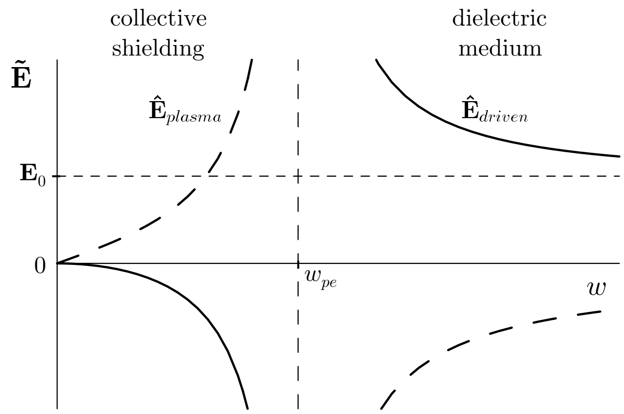

Figure 1.6: Frequency dependence of the electric field components oscillating at the driven frequency ω_0 ($\hat{\mathbf{E}}_{\rm driven}$, solid lines) and the plasma frequency ω_{pe}($\hat{\mathbf{E}}_{\rm plasma}$, dashed lines) induced in a plasma by $\mathbf{E}_{\rm ext} = \mathbf{E}_0 \sin \omega_0 t$, as indicated in Fig. 1.5. The driven frequency response is shielded out for $\omega_0 << \omega_{pe}$; it approaches the imposed electric field for $\omega_0 >> \omega_{pe}$. The plasma frequency response is induced by the process of turning on the external electric field; it becomes small when ω_0 is very different from ω_{pe}. The singular behavior for $\omega_0 \to \omega_{pe}$ results from driving the system at the natural oscillation frequency of the plasma, the plasma frequency; it is limited in more complete plasma models by collisional, kinetic or nonlinear effects.

this estimate we take $\mathbf{v}_0$ to be of order the most probable electron thermal speed $v_{Te} \equiv \sqrt{2T_e/m_e}$ [see (**??**) in Section A.3]. Also, we estimate t by $1/\omega$. However, since the most important plasma effects occur for $\omega \sim \omega_{pe}$ (see Fig. 1.6), we scale ω to ω_{pe}. Then, since $v_{Te}/\omega_{pe} = \sqrt{2}\,\lambda_{De}$, the particle streaming part of (1.35) leads, neglecting numerical factors, to the condition

$$L >> \lambda_{De}\,(\omega_{pe}/\omega)\,. \tag{1.49}$$

That is, for $\omega \sim \omega_{pe}$ the plasma must be large compared to the electron Debye length.

For validity of the nonlinear condition $\tilde{\mathbf{x}} \cdot \boldsymbol{\nabla} \tilde{\mathbf{E}} << \tilde{\mathbf{E}}$ we consider a situation where $\tilde{E} = (\tilde{\Phi}/L) \sin \omega t$. Then, again neglecting numerical factors, we find that to neglect the nonlinearities we must require

$$\frac{e\tilde{\Phi}}{T_e} << \frac{L^2}{\lambda_{De}^2}\left(\frac{\omega^2}{\omega_{pe}^2}\right). \tag{1.50}$$

Since we can anticipate from physical considerations that potential fluctuations $\tilde{\Phi}$ are at most of order some modest factor times the electron temperature in a plasma, the nonlinear criterion is usually well satisfied as long as the streaming criterion in (1.49) is. Hence, our derivation of the plasma polarization is generally valid for $\omega \sim \omega_{pe}$ plasma oscillation phenomena as long as the plasma under consideration is much larger than the electron Debye length.

We can also use the preceding logic to specify the temporal and spatial scales on which the inertial response and effects discussed in this section apply in an infinite, homogeneous plasma — versus the conditions where the adiabatic response in the first section of this chapter apply. (For a general discussion of inertial and adiabatic responses — for a harmonic oscillator — see Appendix E.) For $\omega \sim \omega_{pe}$, as long as the scale length $L \sim \delta x$ of interest is long compared to the electron Debye length λ_{De}, conditions (1.35), (1.49) and (1.50) are all satisfied. Then, the inertial and electron plasma oscillation effects we have discussed are relevant since $\omega >> v_{Te}/\delta x$, which is the inverse of the time required for a thermal electron to move a distance δx. However, for low frequencies $\omega << \omega_{pe}$ such that $\delta x << \lambda_{De}(\omega/\omega_{pe})$, or for scale lengths $\delta x << \lambda_{De}$ with $\omega \sim \omega_{pe}$, the inequality conditions become reversed and the approximations we have used in this section break down. Then, instead of an inertial response, we obtain an adiabatic response for $\omega << v_{Te}/\delta x$ and the Debye shielding effects we discussed in the first section of this chapter. Intermediate situations with $\delta x \sim \lambda_{De}(\omega_{pe}/\omega) \sim v_{Te}/\omega$ must be treated kinetically — see Chapter 8.

1.5 Plasma as a Dielectric Medium

In general, any vector field such as the electric field perturbation $\tilde{\mathbf{E}}$ is composed of both longitudinal (irrotational, $\nabla\cdot\tilde{\mathbf{E}} \neq 0$) and transverse (solenoidal, $\nabla\cdot\tilde{\mathbf{E}} = 0$) parts — see Section D.5 of Appendix D. From the form of (1.43) it is clear that we have been discussing the longitudinal component of the electric field perturbation. This component is derivable from a potential, $\tilde{\mathbf{E}} = -\nabla\tilde{\phi}$, and represents the electrostatic component of the electric field perturbation. Since we have $\nabla\cdot\tilde{\mathbf{E}} = -\nabla^2\tilde{\phi} \neq 0$, we see from Gauss's law (1.1) that these electrostatic perturbation components correspond to charge density perturbations in the plasma. Thus, the electron plasma oscillations we have been discussing are electrostatic "space charge" oscillations in which the longitudinal component of the electric field and plasma polarization oscillate out of phase with respect to each other, i.e., $\partial^2(\nabla\cdot\tilde{\mathbf{E}})/\partial t^2 = -\,\omega_{pe}^2\nabla\cdot\tilde{\mathbf{E}} = -\,\partial^2(\nabla\cdot\tilde{\mathbf{P}})/\partial t^2$.

The polarizability of the plasma by an electric field perturbation can also be interpreted by considering the plasma to be a dielectric medium. To illustrate this viewpoint, we note that in a dielectric medium Gauss's law becomes [see (**??**) in Section A.2]

$$\nabla\cdot\mathbf{D} = \rho_{\text{free}}, \tag{1.51}$$

where

$$\mathbf{D} \equiv \epsilon\,\mathbf{E} \tag{1.52}$$

is the displacement vector, ρ_{free} is the charge density of the free charges (i.e., those not contributing to the plasma dielectric), and ϵ is the dielectric constant of the medium ($\epsilon = \epsilon_0$ for a vacuum). The electric field perturbation $\tilde{\mathbf{E}}$ induces the polarization charge density given in (1.41) and the polarization $\tilde{\mathbf{P}}$. Comparing (1.51) with (1.41) and (1.42), we deduce that the perturbed displacement vector $\tilde{\mathbf{D}}$ is related to the polarization $\tilde{\mathbf{P}}$ through [see (**??**) and (**??**)]

$$\tilde{\mathbf{D}} = \epsilon_0 \tilde{\mathbf{E}} + \tilde{\mathbf{P}} \equiv \epsilon_0 \left(1 + \hat{\chi}_E\right) \tilde{\mathbf{E}} \equiv \hat{\epsilon}\, \tilde{\mathbf{E}}, \tag{1.53}$$

with

$$\tilde{\mathbf{P}} \equiv \epsilon_0 \hat{\chi}_E \tilde{\mathbf{E}} \tag{1.54}$$

in which $\hat{\chi}_E$ is the *electric susceptibility* of the plasma. We have placed hats over ϵ and χ_E to emphasize that these quantities are only defined with respect to temporally (and later spatially) varying electric fields; that is, unlike regular dielectric media, their static, homogeneous plasma limits are divergent and hence do not exist (see below).

For the sinusoidal electric field perturbations of the form $\tilde{\mathbf{E}} = \hat{\mathbf{E}} \sin \omega t$ that we have been discussing, the polarization density $\tilde{\mathbf{P}}$ given by (1.36) becomes

$$\tilde{\mathbf{P}} = -\, \epsilon_0 \sum_s \frac{\omega_{ps}^2}{\omega^2} \tilde{\mathbf{E}} \equiv \epsilon_0 \hat{\chi}_E \tilde{\mathbf{E}}; \tag{1.55}$$

hence, we have

$$\hat{\chi}_E(\omega) = -\sum_s \frac{\omega_{ps}^2}{\omega^2} \simeq -\frac{\omega_{pe}^2}{\omega^2} \tag{1.56}$$

and

$$\boxed{\hat{\epsilon}_I(\omega) = \epsilon_0 \left(1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}\right) \simeq \epsilon_0 \left(1 - \frac{\omega_{pe}^2}{\omega^2}\right), \quad \text{inertial dielectric.}} \tag{1.57}$$

In obtaining this form of $\tilde{\mathbf{P}}$ we have performed the integrals in (1.36) as indefinite integrals in time and hence neglected the initial conditions — because in determining dielectric properties of a medium one considers only the time asymptotic response and neglects the initial transient effects.

The frequency dependence of the *inertial dielectric*[8] $\hat{\epsilon_I}(\omega)$ in (1.57), which represents the inertial response of a plasma, is shown in Fig. 1.7. The fact that $\hat{\epsilon_I}(\omega) \to \epsilon_0$ for $\omega >> \omega_{pe}$ shows why the $\hat{\mathbf{E}}_{\rm driven}$ component in (1.48) approaches the externally applied electric field in this "vacuum" limit. Since $\hat{\epsilon_I}(\omega)$ is negative for $\omega < \omega_{pe}$, the externally applied electric field is shielded

[8]For media such as water the dielectric response function is nearly constant over most relevant frequency ranges, e.g., for visible light. Hence its properties are characterized by a dielectric "constant." However, in plasmas the dielectric response function often varies significantly with frequency (and wavenumber $\mathbf{k}$). Thus, in plasmas we will usually try to avoid speaking of a dielectric "constant;" instead we will just refer to the plasma "dielectric." However, when the dielectric response function is evaluated for a particular frequency (and wavenumber $\mathbf{k}$), we will often call it the dielectric "constant."

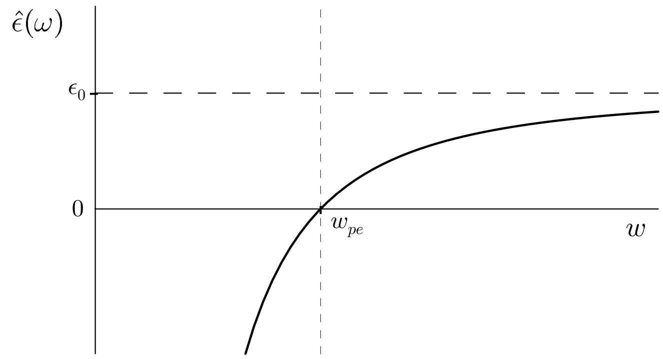

Figure 1.7: Frequency dependence of inertial response plasma dielectric.

out of the plasma or "cut off" in this frequency range. The vanishing of $\hat{\epsilon_I}$ for $\omega = \omega_{pe}$ indicates that this is a "normal mode" of oscillation of the plasma, as is evident from the plasma oscillator equation (1.44) — driven electric fields at frequencies where the dielectric vanishes lead to unbounded resonant responses in linear theory, as can be inferred from (1.51) and (1.52). Also, since $\hat{\epsilon_I}$ is small for ω close to ω_{pe}, the transient plasma frequency response $\hat{\mathbf{E}}_{\rm plasma}$ is largest in this frequency range. Finally, we note that $\hat{\epsilon_I}(\omega)$ is divergent in the $\omega \to 0$ or static limit. Thus, the inertial dielectric response of a plasma can only be defined for temporally varying processes.

Because the polarization of the plasma is 180° out of phase with respect to the electric field perturbations for all real ω, the inertial plasma response is reactive (i.e., not dissipative) for all frequencies ω. That there is no dissipation can be demonstrated explicitly by calculating the average Joule heating $\tilde{\mathbf{E}} \cdot \tilde{\mathbf{J}}$ with $\tilde{\mathbf{J}} = n_0 e \tilde{\mathbf{v}}$ over an oscillation period $2\pi/\omega$ and showing that it vanishes. If dissipative effects, such as collisons, are added, they lead to wave damping through the joule heating they induce in the plasma — see Problem 1.18

The energy density of plasma oscillations is composed of two parts: the vacuum electric field energy density $\epsilon_0 |\tilde{\mathbf{E}}|^2/2$ and the polarization energy density $w_{\rm pol} = -\frac{1}{2}\tilde{\mathbf{P}} \cdot \tilde{\mathbf{E}} = -\frac{1}{2}\epsilon_0 \hat{\chi}_E |\tilde{\mathbf{E}}|^2$. For an electric field perturbation $\tilde{\mathbf{E}}$ oscillating at frequency ω the polarization is given in (1.55). Thus, we find $w_{\rm pol} = (\epsilon_0/2)(\omega_{pe}^2/\omega^2)|\tilde{\mathbf{E}}|^2$. Hence, the total energy density [see (**??**)] in an electrostatic plasma oscillation is given by

$$\boxed{w_E \equiv \frac{1}{2}(\tilde{\mathbf{E}} \cdot \tilde{\mathbf{D}}) = \frac{\epsilon_0}{2}|\tilde{\mathbf{E}}|^2 + w_{\rm pol} = \frac{\epsilon_0}{2}\left(1 + \frac{\omega_{pe}^2}{\omega^2}\right)|\tilde{\mathbf{E}}|^2, \quad \text{wave energy density.}} \tag{1.58}$$

For low frequencies ($\omega << \omega_{pe}$), for which an externally imposed electric field is shielded out of the plasma, the polarization energy density is dominant.

In contrast, for high frequencies ($\omega >> \omega_{pe}$) the electron inertia effects cause the polarization to be small; then, the energy density is predominantly just that residing in the electric field perturbation itself. The fact that the energy density caused by electric field perturbations can have a significant (or even dominant, as occurs for $\omega << \omega_{pe}$) component due to the polarizability of the plasma is a very important aspect of plasma oscillations.

1.6 Ion Acoustic Waves

In the preceding sections we have implicitly assumed that the electrons and ions both exhibit either adiabatic or inertial responses. However, because the ions are much heavier, they have a much lower inertial or plasma frequency and, for the typical case where $T_e \sim T_i$, a much lower thermal speed than electrons. Thus, for a given length scale δx there is an intermediate frequency regime $v_{Ti}/\delta x << \omega << v_{Te}/\delta x$ in which the ions respond inertially while the electrons respond adiabatically. We will now determine the equation governing electric field perturbations in a plasma in this regime.

The perturbed electron density for an adiabatic ($\omega << v_{Te}/\delta x$) response induced by a potential perturbation $\tilde{\phi}$ of a quasineutral plasma equilibrium ($\sum_s n_{0s} q_s = 0$) is obtained from the perturbed Boltzmann relation (1.4):

$$\tilde{n}_e = n_{0e} \frac{e\tilde{\phi}}{T_e}. \tag{1.59}$$

The perturbed ion density for an inertial ($\omega >> v_{Ti}/\delta x$) response induced by an electric field perturbation $\tilde{\mathbf{E}}$ is obtained from the ion polarization part of the total plasma charge density given in (1.41):

$$\tilde{n}_i = -\frac{\epsilon_0}{q_i}\,\omega_{pi}^2 \int_0^t dt' \int_0^{t'} dt'' \;\nabla\cdot\tilde{\mathbf{E}}(t''). \tag{1.60}$$

The overall perturbed charge density in this intermediate frequency regime is thus given by

$$\begin{aligned}\frac{\tilde{\rho}_q}{\epsilon_0} &= \sum_s \frac{\tilde{n}_s q_s}{\epsilon_0} = -\,\omega_{pi}^2 \int_0^t dt' \int_0^{t'} dt'' \;\nabla\cdot\tilde{\mathbf{E}}(t'') - \frac{n_{0e}e^2}{\epsilon_0 T_e}\tilde{\phi} \\ &= +\,\omega_{pi}^2 \int_0^t dt' \int_0^{t'} dt'' \;\nabla^2\tilde{\phi}(t'') - \frac{\tilde{\phi}}{\lambda_{De}^2}\end{aligned} \tag{1.61}$$

in which we have specialized to electrostatic perturbations for which $\tilde{\mathbf{E}} = -\nabla\tilde{\phi}$ and $\nabla\cdot\tilde{\mathbf{E}} = -\nabla^2\tilde{\phi}$.

Substituting this perturbed charge density into Poisson's equation (1.1), we obtain

$$-\left(\nabla^2 - \frac{1}{\lambda_{De}^2}\right)\tilde{\phi} = \omega_{pi}^2 \int_0^t dt' \int_0^{t'} dt'' \;\nabla^2\tilde{\phi}(t'').$$

Or, taking the second partial derivative with respect to time, this yields

$$-\left(\nabla^2 - \frac{1}{\lambda_{De}^2}\right)\frac{\partial^2 \tilde{\phi}}{\partial t^2} = \omega_{pi}^2 \nabla^2 \tilde{\phi}. \qquad (1.62)$$

Considering perturbations whose scale lengths are long compared to the electron Debye length $\left(\nabla^2 << 1/\lambda_{De}^2\right)$, the equation governing potential perturbations in the intermediate frequency regime becomes simply

$$\frac{\partial^2 \tilde{\phi}}{\partial t^2} - c_S^2 \nabla^2 \tilde{\phi} = 0, \quad \text{ion acoustic wave equation,} \qquad (1.63)$$

in which

$$c_S^2 = \omega_{pi}^2\, \lambda_{De}^2 = \frac{T_e}{m_i}\,\frac{n_i q_i^2}{n_e q_e^2} = \frac{Z_i T_e}{m_i}. \qquad (1.64)$$

As indicated in the last equality, for a plasma with a single ion component $n_i q_i^2 = Z_i n_e e^2$ so that $c_S^2 = Z_i T_e/m_i$. The quantity c_S has the units of a speed and as we will see below is the speed of ion acoustic (or sound) waves in a plasma. It is given numerically by

$$\boxed{c_S \equiv \sqrt{\frac{Z_i T_e}{m_i}} \simeq 10^4 \sqrt{\frac{Z_i T_e(\text{eV})}{A_i}}\ \text{m/s}, \quad \text{ion acoustic speed,}} \qquad (1.65)$$

in which A_i is the atomic mass of the ions in the plasma: $A_i \equiv m_i/m_p$. Here, we have used the subscript S on c to indicate that these ion acoustic waves are the natural "sound" (S) waves that occur in a plasma. The relation of ion acoustic waves to normal sound waves in a neutral gas are discussed at the last of this section, and their relation to the sound waves in a magnetohydrodynamic description of a plasma is discussed in Section 7.2.

Equation (1.63) is a wave equation. In one dimension, say the x direction, general solutions of it are given by a linear combination of arbitrary functions f_1, f_2 of its mathematical characteristics $\varphi_\pm \equiv x \mp c_S t$:

$$\tilde{\phi}(x,t) = C_1 f_1(x - c_S t) + C_2 f_2(x + c_S t),$$

where C_1 and C_2 are arbitrary constants to be fixed by the boundary conditions. A point of constant phase in this solution moves at the phase speed V_φ of the wave: $d\varphi_\pm = 0 = dx \mp c_S dt \Longrightarrow V_\varphi = dx/dt = \pm c_S$ along the mathematical characteristics $x = x_0 \pm c_S t$.

For wave-like equations such as those in (1.62) or (1.63) we usually seek solutions of the form

$$\tilde{\phi}\,(\mathbf{x},t) = \hat{\phi}\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \qquad (1.66)$$

in which $\hat{\phi}$ is a constant, $\mathbf{k}$ is the (vector) wavenumber and ω is the frequency of the wave. Substituting this Ansatz (proposed form) into (1.62), we find

$$\left[-\left(-k^2 - 1/\lambda_{De}^2\right)\left(-\omega^2\right) + \omega_{pi}^2 k^2\right]\hat{\phi} = 0.$$

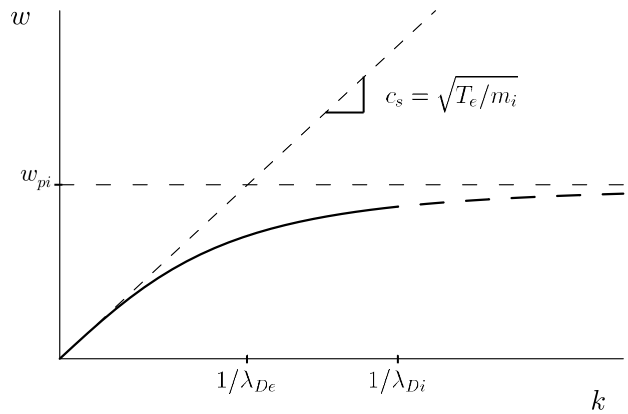

Figure 1.8: Dispersion diagram for ion acoustic waves in an electron-proton plasma with $T_e >> T_i$. For $k\lambda_{Di} << 1$ the ion acoustic waves propagate at the ion acoustic speed: $\omega/k \simeq c_S$. The dispersion curve $\omega = \omega(k)$ is shown as a dashed line for $k\lambda_{Di} \gtrsim 1$ because in this region the ion response is no longer inertial (kinetic effects become important) and the present analysis becomes invalid.

For nontrivial solutions with $\hat{\phi} \neq 0$, we must have

$$\omega^2 = \frac{k^2 c_S^2}{1 + k^2 \lambda_{De}^2} = \frac{\omega_{pi}^2}{1 + 1/\left(k^2 \lambda_{De}^2\right)}, \quad \text{ion acoustic wave dispersion relation.} \tag{1.67}$$

This is called a dispersion relation because it givess the dependence of ω on k — here for electrostatic *ion acoustic waves* propagating in a plasma.

The dispersion diagram (ω versus k) for ion acoustic waves is shown in Fig. 1.8. For $k^2\lambda_{De}^2 << 1$ (long scale lengths compared to the electron Debye length) we have $\omega/k = \pm c_S$ — the phase speed $V_\varphi \equiv \omega/k$ of the wave is the ion acoustic speed c_S. Since we have assumed that the ions have an inertial response, taking $\delta x \sim 1/k$ we must have $v_{Ti}/\delta x \sim k v_{Ti} << \omega \simeq k c_s$. This condition is satisfied and ion acoustic waves exist in an electron-proton plasma only if the ion acoustic speed $c_S \equiv \sqrt{T_e/m_i}$ is much larger than the ion thermal speed $v_{Ti} = \sqrt{2T_i/m_i}$, which occurs only if $T_e >> 2T_i$. As can be discerned from (1.67), the wave frequency ω increases for increasing $k\lambda_{De}$ and asymptotes to ω_{pi} for $k\lambda_{De} >> 1$. However, to satisfy the required condition for an ion inertial response we must have $k v_{Ti} << \omega \sim \omega_{pi}$ or $k\lambda_{Di} << 1$. We can satisfy $k\lambda_{De} >> 1 >> k\lambda_{Di}$ only if $T_e >> T_i$, which is the same as the condition noted previously in this paragraph for the existence of ion acoustic waves.

As we discussed in the preceeding section, plasma responses can also be described terms of the plasma giving a dielectric response $\hat{\epsilon}$. For waves of the form given in (1.66) the polarization corresponding to the perturbed charge density in (1.61) becomes

$$\tilde{\mathbf{P}} = \epsilon_0 \left(-\frac{\omega_{pi}^2}{\omega^2} + \frac{1}{k^2 \lambda_{De}^2} \right) \tilde{\mathbf{E}}, \tag{1.68}$$

in which we have used $\tilde{\mathbf{E}} = -i\mathbf{k}\tilde{\phi}$ and $\boldsymbol{\nabla} \cdot \tilde{\mathbf{E}} = -\nabla^2 \tilde{\phi} = k^2 \tilde{\phi}$. Using the definitions for the interrelationships between $\tilde{\mathbf{P}}, \hat{\chi}_E$ and $\hat{\epsilon}$ given in (1.53), (1.54), we find that in the intermediate frequency regime we are considering the plasma dielectric response is given by

$$\boxed{\hat{\epsilon}_S(\mathbf{k}, \omega) = \epsilon_0 \left(1 - \frac{\omega_{pi}^2}{\omega^2} + \frac{1}{k^2 \lambda_{De}^2} \right), \quad \text{ion acoustic dielectric.}} \tag{1.69}$$

Setting this $\hat{\epsilon}_S$ to zero to obtain the normal modes of the plasma readily yields the dispersion relation for ion acoustic waves given in (1.67). This dielectric function diverges for either $\omega \to 0$ or $k \to 0$. Thus, again, the plasma dielectric is only a meaningful quantity for temporal and spatially varying perturbations, i.e., not for an infinite, homogeneous equilibrium. Also, since $\hat{\epsilon}_S$ is real for all real $\mathbf{k}, \omega$ (i.e.,the electron and ion components of the polarization are in phase or 180° out of phase with the electric field perturbation), this intermediate frequency response is also totally reactive (i.e., not dissipative).

Ion acoustic waves are similar to but somewhat different from ordinary sound waves in a neutral gas. Ordinary sound waves are compressible ($\boldsymbol{\nabla} \cdot \tilde{\mathbf{V}} \neq 0$ where $\tilde{\mathbf{V}}$ is the perturbed flow velocity) mass density perturbations induced by momentum perturbations propagated by the collisionally coupled flow of the neutral gas molecules or atoms in response to pressure perturbations — see Section A.6. They propagate at a "hydrodynamic" (H) phase speed given by $c_S^H = \sqrt{\Gamma p_n / \rho_m} = \sqrt{\Gamma T_n / m_n}$ in which $\Gamma = (N + 2)/N$ is the ratio of the specific heats, N is the number of degrees of freedom, and p_n, ρ_m, T_n and M_n are the neutral gas pressure, mass density, temperature and mass, respectively. In an electron-proton plasma with $T_e >> T_i$, ion acoustic waves propagate via longitudinal ($\boldsymbol{\nabla} \cdot \tilde{\mathbf{E}} \neq 0$) electric field perturbations, which as we will see in Section 7.2 also lead to compressible flow perturbations $\boldsymbol{\nabla} \cdot \tilde{\mathbf{V}} \neq 0$, in which the adiabatic electron polarization charge density is balanced by an inertial ion polarization charge density. Ion acoustic waves propagate at a phase speed $c_S = \sqrt{T_e / m_i}$ with the electron temperature coming from the adiabatic electron Debye shielding and the ion mass coming from the ion inertia. Thus, the physical mechanism responsible for ion acoustic wave propagation in a plasma is different from that of sound waves in a neutral gas even though they are both carried by incompressible flow perturbations — collisions couple the atoms or molecules in a neutral gas whereas the electric field couples electrons and ions together in a plasma. The ion acoustic speed in a $T_e >> T_i$ plasma does not, like ordinary sound waves, depend on the ratio of specific heats or dimensionality of the system — because it is a "one-demensional" electric field perturbation rather than the collisionally-coupled flow in a neutral gas that propagates ion acoustic waves in a plasma.

1.7 Electromagnetic Waves in Plasmas

In the preceding three sections we explored the properties of longitudinal (electrostatic) electric field perturbations in an unmagnetized plasma. In this section we develop the properties of transverse (solenoidal) electric field perturbations for which $\boldsymbol{\nabla}\times\tilde{\mathbf{E}} \neq \mathbf{0}$ but $\boldsymbol{\nabla}\cdot\tilde{\mathbf{E}} = 0$ — see Sections A.2 and D.5. These types of perturbations are often referred to as electromagnetic (em) waves in a plasma and become light waves in the vacuum limit where the plasma effects are negligible.

To investigate electromagnetic waves in a plasma we begin from the two Maxwell equations that involve time-derivatives [see (**??**) in Section A.2]:

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\left(\mathbf{J} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right), \qquad \text{Ampere's law,} \tag{1.70}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \qquad \text{Faraday's law.} \tag{1.71}$$

We combine these equations by taking the partial time derivative of Ampere's law and substitute in $\partial\mathbf{B}/\partial t$ from Faraday's law to obtain

$$-\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}) = \mu_0\frac{\partial\mathbf{J}}{\partial t} + \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}$$

in which we have used the fact that $\mu_0\epsilon_0 = 1/c^2$, where c is the speed of light in a vacuum. Since $-\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}) = \nabla^2\mathbf{E} - \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E})$, for transverse electric fields $\mathbf{E}_t$ ($\boldsymbol{\nabla}\cdot\mathbf{E}_t = 0$) this can be written as

$$\frac{\partial^2\mathbf{E}_t}{\partial t^2} - c^2\nabla^2\mathbf{E}_t = -\frac{1}{\epsilon_0}\frac{\partial\mathbf{J}}{\partial t}. \tag{1.72}$$

This is a wave equation for the transverse electric field $\mathbf{E}_t$. The inhomogeneous term on the right represents the plasma effects. The general Green function solution of this equation, including the plasma inertial response effects, is developed in Problem 1.23.

Because electromagnetic waves in a plasma are relatively fast (high frequency) phenomena, we can anticipate that the plasma response will be inertial. Thus, the current perturbation induced by the effect of an electric field perturbation $\tilde{\mathbf{E}}_t$ on the charged particles in a plasma is given by

$$\tilde{\mathbf{J}} = \sum_s n_{0s} q_s \tilde{\mathbf{v}}_s \tag{1.73}$$

in which $\tilde{\mathbf{v}}_s$ is the particle velocity perturbation given in (1.32). Taking the partial derivative of this current with respect to time, we obtain

$$\frac{1}{\epsilon_0}\frac{\partial\tilde{\mathbf{J}}}{\partial t} = \sum_s \frac{n_{0s} q_s^2}{m_s\epsilon_0}\tilde{\mathbf{E}}_t = \sum_s \omega_{ps}^2\tilde{\mathbf{E}}_t \simeq \omega_{pe}^2\tilde{\mathbf{E}}_t. \tag{1.74}$$

[This result can also be obtained by considering the plasma to be a dielectric medium with the inertial dielectric given by (1.57) and calculating the time derivative of the displacement current and subtracting off the vacuum contribution: $(1/\epsilon_0)\partial^2\mathbf{D}_t/\partial t^2 - \partial^2\mathbf{E}_t/\partial t^2 \;=\; -\omega^2(\hat{\epsilon}_I/\epsilon_0 - 1)\tilde{\mathbf{E}}_t = \omega_{pe}^2\tilde{\mathbf{E}}_t$.]

Substituting the resultant inertial plasma response into (1.72), we obtain

$$\frac{\partial^2\tilde{\mathbf{E}}_t}{\partial t^2} + \omega_{pe}^2\tilde{\mathbf{E}}_t - c^2\nabla^2\tilde{\mathbf{E}}_t = \mathbf{0}. \tag{1.75}$$

This equation is the same as (1.44), which we obtained for electrostatic (or longitudinal electric field) perturbations, except for the presence of the $c^2\nabla^2\tilde{\mathbf{E}}_t$ term, which leads to light wave solutions for $\omega_{pe}^2 \to 0$. Thus, (1.75) embodies a combination of charged particle inertial (plasma frequency) and light wave effects in a plasma.

To explore the properties of electromagnetic waves in a plasma we consider wave solutions of the form

$$\tilde{\mathbf{E}}_t(\mathbf{x}, t) = \hat{\mathbf{E}}_t\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}. \tag{1.76}$$

Substituting this Ansatz (proposed form) for $\tilde{\mathbf{E}}_t$ into (1.75) yields

$$\left(-\omega^2 + \omega_{pe}^2 + k^2c^2\right)\hat{\mathbf{E}}_t = \mathbf{0}.$$

Nontrivial ($\hat{\mathbf{E}}_t \neq \mathbf{0}$) solutions are possible for electromagnetic waves that satisfy

$$\boxed{\omega^2 = \omega_{pe}^2 + c^2k^2, \quad \text{or} \quad k = \pm\sqrt{\omega^2 - \omega_{pe}^2}\,/\,c, \quad \text{em wave dispersion relation.}} \tag{1.77}$$

This dispersion relation is plotted in Fig. 1.9. Since for these waves ω/k is greater than the speed of light and hence, for a nonrelativistic plasma, the thermal speeds of both the ions and electrons, it was valid for us to use the inertial response that we utilized in (1.74). In the short wavelength limit ($k >> c/\omega_{pe}$) the inertial plasma effects become negligible and we have regular light waves with $\omega \simeq \pm ck$. For longer wavelengths ($k \lesssim \omega_{pe}/c$), but still high enough frequency so that $\omega > \omega_{pe}$, the waves have the dispersion characteristics shown in Fig. 1.9. For $\omega_{pe}/c >> k$, the waves become *electromagnetic plasma oscillations* with $\omega \simeq \omega_{pe}$. For $\omega < \omega_{pe}$, the wavenumber k becomes imaginary; this indicates that transverse electric field perturbations are spatially evanescent in this regime. In the limit $\omega << \omega_{pe}$ we have $k \simeq \pm\, i\, \omega_{pe}/c$.

To make the properties of electromagnetic waves in a plasma more concrete, we consider the propagation of electromagnetic waves from a vacuum into a plasma. As shown in Fig. 1.10, we consider a situation in which the infinite half-space where $x > 0$ is filled with plasma while the infinite half-space where $x < 0$ is a vacuum. A wave of frequency ω is launched from $x = -\infty$ in the $+x$ direction toward the plasma and is incident (I) on the plasma at $x = 0$. It will

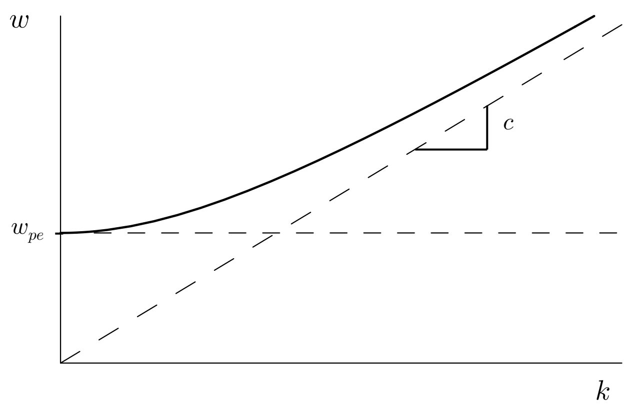

Figure 1.9: Dispersion diagram for electromagnetic waves in a plasma. For $kc << \omega_{pe}$ the waves become electromagnetic plasma oscillations with $\omega \simeq \omega_{pe}$. For $kc >> \omega_{pe}$ the waves become ordinary light waves with $\omega \simeq ck$.

be taken to be of the form:

$$\text{incident wave:} \quad \tilde{\mathbf{E}}_t = \hat{\mathbf{E}}_I e^{i\mathbf{k}_I\cdot\mathbf{x}-i\omega t}, \quad \hat{\mathbf{E}}_I = \hat{E}_I\hat{\mathbf{e}}_{\rm y}, \quad \mathbf{k}_I = k_0\hat{\mathbf{e}}_{\rm x}, \quad k_0 = \omega/c \tag{1.78}$$

in which for simplicity we have assumed that the incident wave has linear polarization in the y direction.

In general, part of this wave will be transmitted into the plasma at the vacuum-plasma interface at $x = 0$. We take the transmitted (T) wave to be of the form

$$\text{transmitted wave:} \quad \tilde{\mathbf{E}}_t = \hat{\mathbf{E}}_T e^{i\mathbf{k}_T\cdot\mathbf{x}-i\omega t}, \quad \hat{\mathbf{E}}_T = \hat{E}_T\hat{\mathbf{e}}_{\rm y}, \quad \mathbf{k}_T = k_T\hat{\mathbf{e}}_{\rm x}, \quad k_T = \sqrt{\omega^2-\omega_{pe}^2}\,/\,c \tag{1.79}$$

in which the polarization has again been taken to be in the y direction because the presence of the plasma does not change the wave polarization. In addition, part of the wave will be reflected; we take the reflected (R) wave to be of the form

$$\text{reflected wave:} \quad \tilde{\mathbf{E}}_t = \hat{\mathbf{E}}_R e^{i\mathbf{k}_R\cdot\mathbf{x}-i\omega t}, \quad \hat{\mathbf{E}}_R = \hat{E}_R\hat{\mathbf{e}}_{\rm y}, \quad \mathbf{k}_R = -k_0\hat{\mathbf{e}}_{\rm x}, \quad k_0 = \omega/c. \tag{1.80}$$

The magnetic field accompanying each of these waves is obtained from Faraday's law (1.71) for wave solutions of the form (1.76): $i\omega\tilde{\mathbf{B}} = i\mathbf{k}\times\tilde{\mathbf{E}}_t \implies \tilde{B}_z = \hat{\mathbf{e}}_{\rm z}\cdot(\mathbf{k}\times\hat{\mathbf{e}}_{\rm y})\,\tilde{E}_y/\omega = k\tilde{E}_y/\omega$. The boundary conditions at the vacuum-plasma interface ($x = 0$) are that the electric field $\tilde{E}_y$ and magnetic field $\tilde{B}_z$ must be

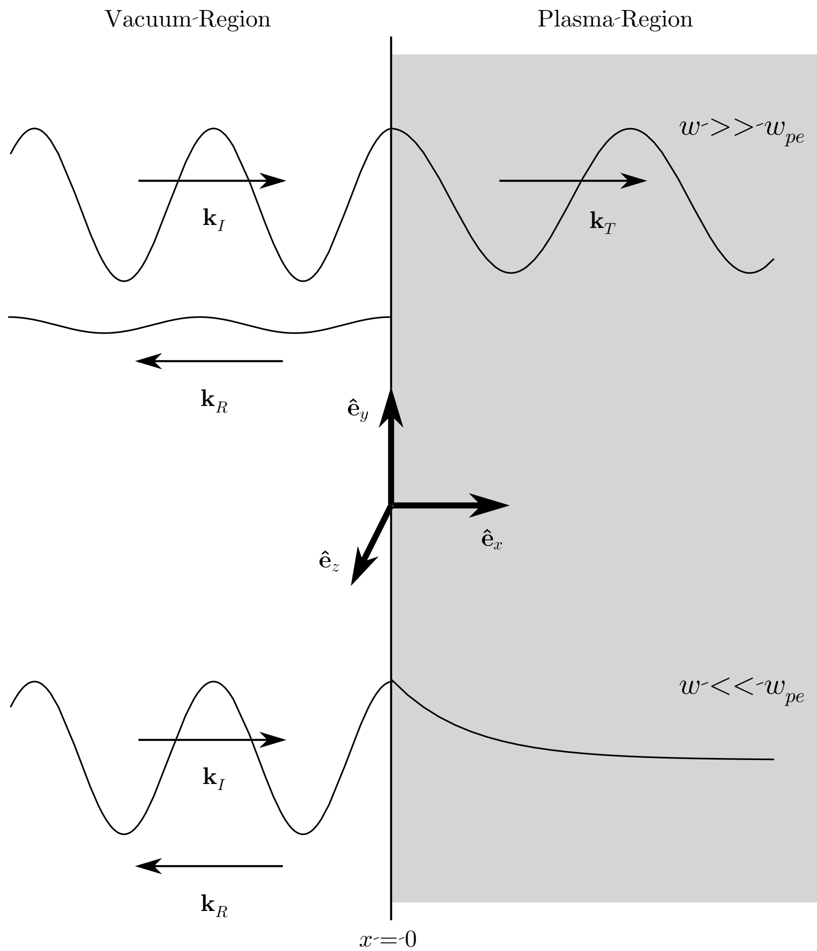

Figure 1.10: Propagation of an incident (I) electromagnetic wave from a vacuum into a plasma. For $\omega >> \omega_{pe}$ the wave is transmitted (T) into the plasma with little reflection (R); the wavenumber k is reduced from ω/c in the vacuum to $(\omega^2 - \omega_{pe}^2)^{1/2}/c$ in the plasma. For $\omega << \omega_{pe}$ the wave is mostly reflected from the plasma; the part that does penetrate into the plasma is exponentially evanescent in the electromagnetic skin depth distance c/ω_{pe}

.

continuous there. They lead to the two conditions

$$\hat{E}_I + \hat{E}_R = \hat{E}_T,$$

$$(k_0/\omega)\left(\hat{E}_I - \hat{E}_R\right) = (k_T/\omega)\,\hat{E}_T.$$

Solving these equations for the relative magnitudes of the transmitted and reflected waves, we find

$$\text{transmitted:}\qquad \frac{\hat{E}_T}{\hat{E}_I} = \frac{2k_0}{k_0 + k_T} = \frac{2\omega}{\omega + \sqrt{\omega^2 - \omega_{pe}^2}},$$

$$\text{reflected:}\qquad \frac{\hat{E}_R}{\hat{E}_I} = \frac{k_0 - k_T}{k_0 + k_T} = \frac{\omega - \sqrt{\omega^2 - \omega_{pe}^2}}{\omega + \sqrt{\omega^2 - \omega_{pe}^2}}.$$

The properties of the transmitted and reflected electromagnetic waves are shown in Fig. 1.10 for two extreme limits: $\omega >> \omega_{pe}$ and $\omega << \omega_{pe}$. For very high frequencies ($\omega >> \omega_{pe}$) the incident electromagnetic is transmitted into the plasma with very little reflection and only a slight reduction in the wavenumber k. As the frequency of the incident wave is decreased, the wavenumber decreases to $k_T \equiv \sqrt{\omega^2 - \omega_{pe}^2}/c$. At the point where $\omega = \omega_{pe}$, the transmitted wave has $k_T = 0$ and becomes just an electromagnetic plasma oscillation. For $\omega < \omega_{pe}$ the wavenumber becomes imaginary, $k_T = \pm i\sqrt{\omega_{pe}^2 - \omega^2}/c$. The plus sign is the physically relevant solution since it leads to evanescence (spatial decay not due to a dissipative process) in space for $x > 0$. In the limit $\omega << \omega_{pe}$ the incident wave is mostly reflected and the small component of the wave that is transmitted into the plasma is given by

$$\tilde{\mathbf{E}}_t \simeq \hat{\mathbf{E}}_T \exp[-x/(c/\omega_{pe}) - i\omega t], \quad \omega << \omega_{pe}. \tag{1.81}$$

This electric field perturbation is exponentially evanescent in the distance δ_e given by

$$\boxed{\delta_e \equiv c/\omega_{pe}, \quad \text{electromagnetic skin depth.}} \tag{1.82}$$

Thus, for $\omega > \omega_{pe}$ electromagnetic waves are partially reflected at the vacuum-plasma interface and propagate into plasmas with some reduction in the wavenumber k. However, for $\omega < \omega_{pe}$ the plasma (and in particular the electron) inertial response to an electromagnetic wave causes the wave to be mostly reflected at the vacuum-plasma interface and prevents the wave from penetrating into a plasma more than a distance of about $c/\sqrt{\omega_{pe}^2 - \omega^2}$, which becomes just the electromagnetic skin depth c/ω_{pe} in the limit $\omega << \omega_{pe}$.

A major diagnostic application of the properties of electromagnetic waves in a plasma is their use in a microwave interferometer to determine the density of

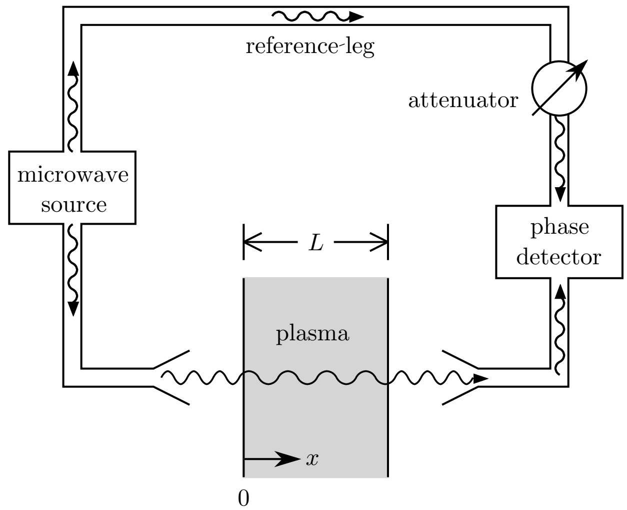

Figure 1.11: Schematic illustration of a microwave interferometer. The electromagnetic wave passing through the plasma has a smaller wavenumber (longer wavelength) than the wave passing through vacuum in the reference leg. Thus, there is a phase shift between the two signals arriving at the detector.

a plasma — see Fig. 1.11. The difference in the phase between the wave that passes through a reference vacuum leg versus the wave that passes through a leg with a plasma in it is given by

$$\Delta\varphi = \int_0^L dx\,[k_I - k_T(x)] = \int_0^L dx \left[\frac{\omega - \sqrt{\omega^2 - \omega_{pe}^2(x)}}{c}\right].$$

In the limit $\omega >> \omega_{pe}$ this becomes simply

$$\Delta\varphi \simeq \int_0^L dx\,\frac{\omega_{pe}^2(x)}{2\omega c} = \frac{e^2}{2\omega m_e \epsilon_0 c}\int_0^L dx\; n_e(x). \tag{1.83}$$

Since the square of the electron plasma frequency is proportional to the local plasma density, the measurement of this phase shift determines the line integral of the electron density in the plasma. For example, microwave interferometers with frequencies in the $50 - 200$ GHz range are commonly used to measure the "line-average" density $\bar{n}_e \equiv (1/L)\int_0^L dx\, n_e(x)$ of plasmas with electron densities in the $10^{18} - 10^{20}$ m^{-3} range. For some other applications in which the properties of electromagnetic waves in plasmas are important see Problems 1.27–1.29.

1.8 Plasma Definition and Responses

Now that we have elucidated the basic length (Debye length λ_D) and time (plasma period $1/\omega_{pe}$) scales for collective phenomena in plasmas, we can specify quantitatively the criteria that must be satisfied for matter to exist in the plasma state. As we discussed in the introduction to this chapter, a general criterion for the existence of a plasma is that charged particle interactions be predominantly collective rather than binary in the medium. For this general criterion to be satisfied, we must require that it be satisfied in charged-particle interactions, as well on the relevant length and time scales for collective phenomena:

1. $n\lambda_D^3 >> 1$. The number of charged particles within a Debye cube (or sphere) must be large so that: a) collective interactions dominate over binary interactions at the mean interparticle separation distance; b) the energy density embodied in the polarization electric field around a given charged particle is small compared to a typical particle's kinetic energy; and c) the thermal noise level is small — see (1.11), (1.14) and (1.15), respectively.

2. $L >> \lambda_D$. The spatial extent of a collection of charged particles must be large compared to the collective interaction scale length for plasmas, the Debye length λ_D, so that: a) the collective interactions are dominated by bulk plasma rather than boundary effects; and b) inertial effects are determined locally — see (1.27) and Fig. 1.3, and (1.49), respectively.

3. $\omega_{pe} >> \nu_{en}$. The collective inertial response frequency in a plasma, the electron plasma frequency ω_{pe}, must be large compared to the electron-neutral collision frequency ν_{en}, so that the fundamental inertial responses, the electrostatic electron plasma oscillations in (1.45) and the plasma oscillation effects on electromagnetic waves in (1.75), are not damped by dissipative neutral particle collision effects.

While we have derived the basic collective phenomena in an unmagnetized plasma, the same physical phenomena occur in magnetized plasmas (primarily along the magnetic field direction); hence these criteria for the existence of the plasma state apply to magnetized plasmas as well.

Among the three criteria for existence of the plasma state, the first one, the requirement that there are many charged particles in a Debye cube, is the necessary condition and the most critical. After this fundamental criterion is satisfied, the second and third criteria are just checks (sufficient conditions) that the behavior of the medium will be dominated by collective plasma phenomena on the basic plasma length and time scales. The fundamental plasma parameter, the number of charged particles in a Debye cube, depends on the plasma temperature and charged-particle density, i.e., $n\lambda_D^3 \propto T^{3/2}/n^{1/2}$. Thus, as shown in Fig. 1.12, we can exhibit the various types of plasmas that occur in nature by showing where they lie relative to lines of constant $n\lambda_D^3$ in a plot of electron temperature versus electron plasma density. As shown in this figure,

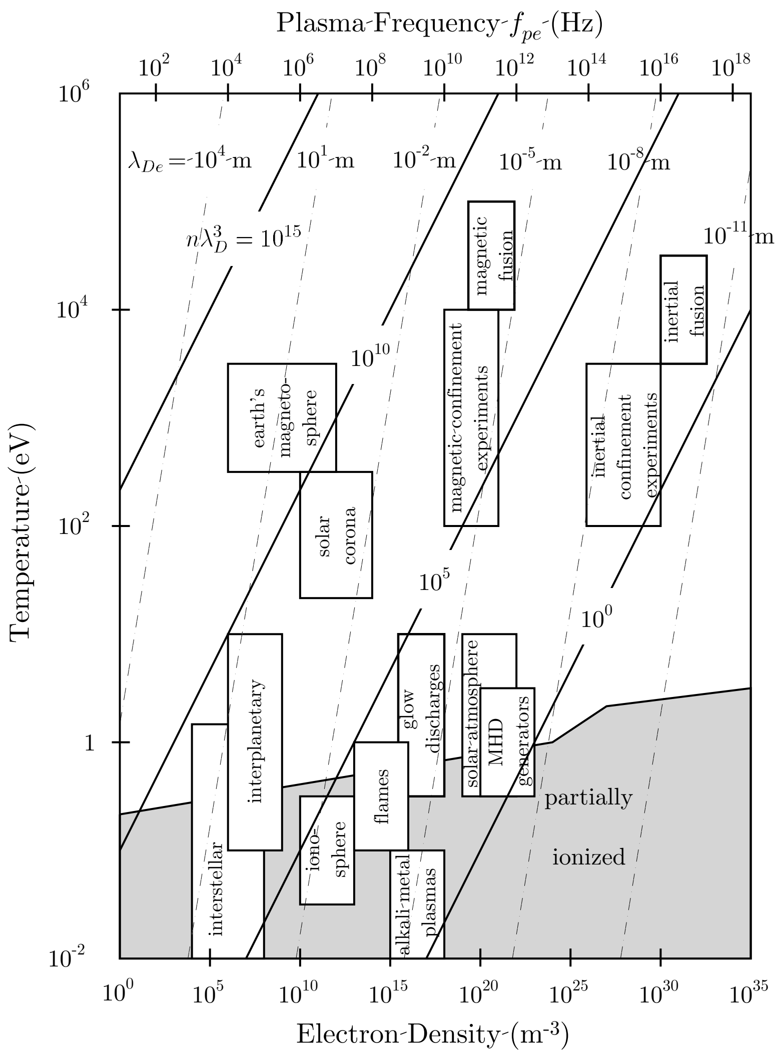

Figure 1.12: Ranges of electron temperature and density for various types of laboratory and extraterrestial plasmas. Also shown are the characteristic plasma parameters: electron Debye length λ_{De} (constant along the dashed lines), number of charged particles in a Debye cube $n_e\lambda_{De}^3$ (constant along solid lines), and electron plasma frequency ω_{pe} (constant along vertical lines). Also indicated is the electron temperature range below which the medium is not fully ionized, which is determined from the Saha equation [see Section A.7 and in particular (**??**) and (**??**)].

the plasma state spans an enormous parameter range — 26 orders of magnitude in density and 7 orders of magnitude in temperature!

Almost all plasmas are electrically *quasineutral* (i.e., $\rho_q = \sum_s n_s q_s \simeq 0$) on length scales longer than the Debye length λ_D. (Notable exceptions are the electron-rich, non-neutral magnetized plasmas — see references listed below.) On length scales shorter than a Debye length the local charge density and potentials are dominated by the effects of the discrete charged particles. However, on length scales longer than the Debye length the collective, plasma polarization effects dominate and the plasma is quasineutral. The use of a quasineutral approximation for scale lengths longer than a Debye length is often called the "plasma approximation."

Slow processes ($\omega << v_T/\delta x \sim k v_T$) in a plasma are governed by the adiabatic response, as discussed in Section 1.1. Fast processes ($\omega >> v_T/\delta x \sim k v_T$) are governed by the inertial response, as discussed in Section 1.4. Because the electron thermal speed is usually much greater than the ion thermal speed ($v_{Te} \sim \sqrt{m_i/m_e}\, v_{Ti} \sim 43\, v_{Ti} >> v_{Ti}$ for $T_e \sim T_i$), the electrons and ions in a plasma can respond differently to perturbations — for $k v_{Ti} << \omega << k v_{Te}$ electrons respond adiabatically while the ions respond inertially, as discussed in Section 1.4.

The response of the plasma to electric field perturbations leads to polarization of the plasma, and hence to a dielectric response for the plasma medium. The plasma responses in the various frequency regimes can be summarized in terms of the density and dielectric responses to small (i.e., linearizable) wavelike perturbations of the form $\exp(i\mathbf{k\cdot x} - i\omega t)$ in an infinite, homogeneous electron-ion plasma as follows:

Adiabatic (A) electrons and ions: $\omega \ll k v_{Ti}, k v_{Te}$; Debye shielding;

$$\tilde{n}_A \simeq -\frac{q\tilde{\phi}}{T} n_0; \quad \hat{\epsilon}_A(\mathbf{k}, \omega) \simeq \epsilon_0 \left[1 + \frac{1}{k^2}\left(\frac{1}{\lambda_{De}^2} + \frac{1}{\lambda_{Di}^2}\right)\right]. \tag{1.84}$$

Adiabatic electrons, inertial ions: $k v_{Ti} \ll \omega \ll k v_{Te}$; ion acoustic waves;

$$\tilde{n}_{eA} \simeq \frac{e\tilde{\phi}}{T_e} n_{0e}, \quad \frac{Z_i e \tilde{n}_{iI}}{\epsilon_0} \simeq -\frac{\omega_{pi}^2}{\omega^2}\boldsymbol{\nabla}\cdot\tilde{\mathbf{E}}; \quad \hat{\epsilon}_S(\mathbf{k}, \omega) \simeq \epsilon_0 \left[1 + \frac{1}{k^2 \lambda_{De}^2} - \frac{\omega_{pi}^2}{\omega^2}\right]. \tag{1.85}$$

Inertial (I) electrons and ions: $k v_{Ti}, k v_{Te} \ll \omega$; plasma oscillations;

$$\frac{q_s \tilde{n}_{sI}}{\epsilon_0} \simeq -\frac{\omega_{ps}^2}{\omega^2}\boldsymbol{\nabla}\cdot\tilde{\mathbf{E}}; \quad \hat{\epsilon}_I(\mathbf{k}, \omega) \simeq \left[1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2}\right]. \tag{1.86}$$

As can be seen from these various responses, a plasma is an electrically active medium with a frequency- and wavenumber-dependent polarizability and dielectric response function. As discussed before, these responses are only applicable for spatially and temporally varying perturbations — they all diverge for $\omega, \mathbf{k} \to 0$.

Within the approximations employed in this chapter, all the basic phenomena in plasmas that we have discussed are reactive with no dissipation. Dissipation would be caused by polarization components that are 90° out of phase with the electric field perturbations, which for $\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$ perturbations would be indicated by an imaginary part of the dielectric $\hat{\epsilon}$.

Implicitly, we have been considering the plasma to be "collisionless." Presuming collisions with neutrals are negligible ($\omega_p >> \nu_n$), there are two types of effects that lead to evanescence of waves in a plasma — Coulomb collisions, which will be discussed in Chapter 2, and wave-particle resonance effects (Landau damping), which will be discussed in Chapter 8. Since the thermal noise fluctuation energy induced by two-particle correlations (or Coulomb collisions) in a plasma is only a small fraction $1/\left(n\lambda_D^3\right) << 1$ of the thermal energy in a plasma, we can anticipate that the average Coulomb collision frequency will also be small: $\nu \sim \omega_p/\left(n\lambda_D^3\right) << \omega_p$. Cumulative small-angle Coulomb collisions enhance the Coulomb collision rate ν by a factor of order $\ln\left(n\lambda_D^3\right)$ — see Chapter 2 — but do not change the basic conclusion that the Coulomb collision rate is slow in a plasma as long as $n\lambda_D^3 >> 1$. The wave-particle resonance effects will be largest when $\omega \sim kv_T$, i.e., when the wave phase speed ω/k is of order the most probable thermal speed $v_T \equiv \sqrt{2T/m}$ of one of the species of charged particles in a plasma. Thus, wave-particle resonance effects will lead to evanescence of waves (Landau damping) for $\omega/k \sim v_{Ti}$ or v_{Te}. These wave phase speeds and corresponding frequencies are between the frequency ranges we have considered in this chapter and require a kinetic plasma description. Wave-particle resonance effects and Landau damping are discussed in Chapter 8, and in particular in Section 8.2.

REFERENCES AND SUGGESTED READING

Discussions of plasma sheath and Langmuir probe theory can be found in

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F.F. Chen, "Electrical Probes," in *Plasma Diagnostic Techniques*, R.H. Huddlestone and S.L. Leonard, eds. (1965), Chapt. 4 [**?**].

L. Schott and R.L.F. Boyd in *Plasma Diagnostics*, W. Lochte-Holtgreven, ed. (1968) [**?**].

J.D. Swift and M.J.R. Schwar, *Electric Probes for Plasma Diagnostics* (1971) [**?**].

P.M. Chung, L. Talbot and K.J. Touryan, *Electric Probes in Stationary and Flowing Plasmas* (1975) [**?**].

P.C. Stangeby, "The Plasma Sheath," in *Physics of Plasma-Wall Interactions in Controlled Fusion*, D.E. Post and R. Behrisch, eds. (1985), Vol. 131, p. 41 [**?**].

Hutchinson, *Principles of Plasma Diagnostics* (1987), Chapt. 3 [**?**].

N. Hershkowitz, "How Langmuir Probes Work," in *Plasma Diagnostics*, O. Auciello and D.L. Flamm, eds., (1990) [**?**].

Recent books that discuss the various types of plasmas indicated in Fig. 1.12, some of which are beyond the scope of this book, include

Magnetically confined plasmas for controlled fusion:

Rose and Clark, Jr., *Plasmas and Controlled Fusion* (1961) [?].

Miyamoto, *Plasma Physics for Nuclear Fusion* (1980) [?].

Miyamoto, *Fundamentals of Plasma Physics and Controlled Fusion* (1997) [?].

Stacey, *Fusion Plasma Analysis* (1981) [?].

Teller, ed., *Fusion* (1981), Vol. I, Parts A & B [?].

Nishikawa and Wakatani, *Plasma Physics, Basic Theory with Fusion Applications* (1990) [?].

Hazeltine and Meiss, *Plasma Confinement* (1992) [?].

White, *Theory of Tokamak Plasmas* (1989) [?].

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Parker, *Cosmical Magnetic Fields, Their Origin And Their Activity* (1979) [?].

Sturrock, *Plasma Physics, An introduction to the theory of astrophysical, geophysical, and laboratory plasmas* (1994) [?].

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Partially ionized plasmas and plasma processing:

Chapman, *Glow Discharge Processes* (1980) [?].

Lieberman and Lichtenberg, *Principles of Plasma Discharges and Materials Processing* (1994) [?].

Nonneutral plasmas:

Davidson, *Physics of Nonneutral Plasmas* (1990) [?].

Marshall, *Free Electron Lasers* (1985) [?].

PROBLEMS

1.1 Evaluate the scale lengths $b_{\rm min}^{\rm cl}$, $n_e^{-1/3}$ and λ_{De} for an electron-proton plasma in a typical small-scale magnetic confinement experiment (e.g., a university-based tokamak) with $n_e = 2 \times 10^{19}$ m^{-3}, $T_e = T_i = 300$ eV. Compare them to the Bohr radius, de Broglie wavelength, and the classical electron radius $r_e = e^2/(\{4\pi\epsilon_0\}m_e c^2)$. Discuss the physical significance of each of these scale lengths. Over what length scale will collective effects occur in such a plasma? /

1.2 Calculate the plasma parameter $n\lambda_D^3$ for the plasma described in the preceding problem. Estimate the effective temperature for thermal noise in such a plasma. Compare this thermal noise temperature to normal room temperature. /

1.3 Consider a hypothetical situation in which all the electrons in a homogeneous and quasineutral but bounded plasma are displaced a small distance x in the $\hat{\mathbf{e}}_{\rm x}$ direction. Show that in the bulk of the plasma the electric field is unchanged, but that in a layer of width x at the plasma edge there is an electric field. How large a displacement x induces a maximum potential change equal to the electron temperature in the plasma? Compare this length to the electron Debye length λ_{De} and discuss why such a comparison is relevant. //

1.4 Determine the one-dimensional potential distribution in a plasma around an infinite sheet charge with a one-dimensional surface "test" charge density given by $\rho_q = \sigma_t \delta(x - x_t)$. ///

1.5 Show that for a two-dimensional situation of an appropriately modified form of (1.7) the potential around a line charge in a plasma is given by

$$\phi_t(\mathbf{x}) = (2\lambda_t/\{4\pi\epsilon_0\})K_0(|\mathbf{x} - \mathbf{x}_t|/\lambda_D)$$

in which λ_t is the line charge density (coulombs/m) for a line charge of infinite length placed at $\mathbf{x} = \mathbf{x}_t$ and K_0 is the modified Bessel function of the second kind of order zero. ///

1.6 Show that the potential given by (1.10) is the Green function for the adiabatic (Debye shielding) response to a free charge density $\rho_{\rm free}(\mathbf{x})$ in an infinite, homogeneous plasma, and thus that the general potential solution is given by

$$\phi(\mathbf{x}) = \int d^3x' \, \frac{\rho_{\rm free}(\mathbf{x}') \exp(-|\mathbf{x} - \mathbf{x}'|/\lambda_D)}{\{4\pi\epsilon_0\} \, |\mathbf{x} - \mathbf{x}'|}.$$

Discuss the physical scale lengths over which this Green's function solution is valid. Compare this result to the corresponding potential induced by a charge density in vacuum given in (**??**). ///

1.7 Show that the combination of the charge of a test particle and the polarization charge density it induces produces a vanishing net charge Q in the plasma. //

1.8 A spherical spacecraft orbiting the earth in a geostationary orbit finds itself immersed in a plasma that typically has an electron density of about 10^6 m^{-3} and temperature of about 100 eV. Sketch the spatial variation of the electric potential around the spacecraft, indicating the magnitudes of the potential and spatial scale lengths involved. To what potential does the spacecraft charge up relative to its surroundings? /

1.9 A spherical probe 3 mm in diameter is inserted into a fully ionized electron-proton plasma generated by microwave heating power applied to a hydrogen gas and has $n_e = 10^{15}$ m^{-3}, $T_e = 10$ eV, $T_i = 1$ eV. What is the Debye length in such a plasma? Is it large or small compared to the probe size? If the probe is biased to -10 V, how much current would it draw? /*

1.10 It is proposed to put a wire screen into the plasma described in the preceding problem and bias it so as to exclude plasma from the region behind the screen. Taking account of sheath effects, how closely spaced must the screen wires be so as to block a substantial fraction of the plasma? To what potential should the probe be biased? /*

1.11 The electron temperature in low energy density plasmas can be measured with a "double probe" — a single assembly with two identical but differentially biased Langmuir probes that electrically floats and draws no net current from the plasma. Show that the current flowing between the two probes is given by

$$I = 2\, I_{Si} \tanh\left(e\Delta\Phi/2T_e\right)$$

in which $\Delta\Phi$ is the potential difference (voltage) between the two probes. //*

1.12 The "static" electrical admittance Y (inverse of impedance Z) of a sheath is given by $\partial I/\partial\Phi_B$. At what bias potential Φ_B should this partial derivative be evaluated? Show that the sheath admittance is given approximately by $Y \simeq I/(T_e/e)$. Up to what frequency will this estimate be valid? //*

1.13 For large negative "wall" potentials ($|\Phi_W| >> T_e/e$) applied between two grids in a planar diode the electrical current is limited by space charge effects. Derive the Child-Langmuir law for this limiting current for a grid separation d as follows. First, using (1.27) show that when the sheath thickness $x_S \to d$ and $\Phi_\infty \to \Phi_W$, the ion speed V_∞ that corresponds to the space-charge-limited ion flow at the sheath edge can be written in terms of the wall potential Φ_W and the grid separation d. Then, show that the (ion) current density into the sheath region between the grids is given by

$$J = \frac{4}{9}\sqrt{\frac{2e}{m_i}}\frac{\epsilon_0|\Phi_W|^{3/2}}{d^2} = (n_e e V_W)\left(\frac{4}{9}\right)\left(\frac{\lambda_{D\Phi}^2}{d^2}\right)$$

in which $V_W \equiv (2e|\Phi_W|/m_i)^{1/2}$ is the ion speed at the wall, and $\lambda_{D\Phi} = (\epsilon_0|\Phi_W|/n_e e)^{1/2}$ is an effective Debye length. //*

1.14 In the Plasma Source Ion Implantation (PSII) technique [J.R. Conrad, J.L. Radtke, R.A. Dodd, F.J. Worzola, N.C. Tran, *J. Appl. Phys.* **62**, 4591 (1987)], the target to be bombarded is inserted into a plasma with parameters $n_e \sim 10^9$cm^{-3} and $T_e \sim 2$ eV, and a natural sheath is allowed to form around it. Then, the target is rapidly biased to a very large (≥ 30 kV) negative potential Φ_B. This expels the lighter electrons from the region around the object, which in turn causes an "ion matrix" to be formed there. On what time scales are the electrons expelled, and the new sheath formed? What is the approximate maximum energy and current density of the ions bombarding the target before the new sheath forms? Compare this current density to that given by the Child-Langmuir law discussed in the preceding problem. Finally, estimate the fluence (ions/cm^2) per pulse and the number of pulses required to inject an atomic monolayer of ions in the target. ///*

1.15 Consider an impure laboratory plasma composed of a number of different types of ions: protons with $n_p = 3\ \times 10^{19}$ m^{-3}, fully ionized carbon ions with 10% of the proton density and iron ions that are 23 times ionized (Lithium-like charge states) with 1% of the proton density. What is the electron density and the overall plasma frequency in this plasma? Also, what is the dielectric "constant" for 90 GHz electrostatic fluctuations in this plasma? /

1.16 Consider electrostatic plasma oscillations in an electron-positron plasma, such as could occur in interstellar space, with $n_{e^-} = n_{e^+} = 10^6$ m^{-3}. What is the plasma frequency for such oscillations? Assume the electrons and positrons have temperatures of 100 eV and that the cross-section for an annihilation interaction between them is given by π times the square of the classical electron radius $r_e = e^2/(\{4\pi\epsilon_0\}m_e c^2)$, i.e., $\sigma_{\rm an} \sim \pi\, r_e^2$. What is their annihilation reaction rate? Compare this annihilation rate to the plasma frequency. /

1.17 Consider a situation where an oscillating potential is applied across two plates on either side of a plasma such as that described in Problem 1.9. Assume the plates are separated by 10 cm and that the potential oscillates at 100 kHz. What is the dielectric "constant" for these oscillations? What is the ratio of the energy density in the plasma polarization fluctuations to that in the electric field fluctuations? /

1.18 Calculate the weak dissipation induced by Coulomb collisions ($\nu << \omega_{pe}$) of $\omega \sim \omega_{pe}$ electrostatic oscillations in a plasma as follows. Add a collisional dynamical friction force $-\,\nu\, m_e \mathbf{v}$ [cf., (**??**)] to (1.31) and show that for $\tilde{\mathbf{E}} = \hat{\mathbf{E}} \sin\omega t$ the perturbed velocity of electrons is then given for $t >> 1/\nu$ by

$$\mathbf{v} \simeq \frac{e\hat{\mathbf{E}}}{\omega m_e}[\cos\omega t - (\nu/\omega)\sin\omega t].$$

Next, calculate the average of the Joule heating in the plasma by the oscillations over an oscillation period $2\pi/\omega$, i.e., $\langle \tilde{\mathbf{J}} \cdot \tilde{\mathbf{E}} \rangle_{\omega t}$. Finally, use a wave energy balance equation [cf., (**??**)] to show that

$$\frac{1}{\langle w_E \rangle_{\omega t}} \frac{\partial \langle w_E \rangle_{\omega t}}{\partial t} = -\,\frac{\langle \tilde{\mathbf{J}} \cdot \tilde{\mathbf{E}} \rangle_{\omega t}}{\langle w_E \rangle_{\omega t}} \simeq -\,\frac{2\nu}{1 + \omega^2/\omega_{pe}^2}$$

in which $\langle w_E \rangle_{\omega t}$ is the average of the electrostatic wave energy density in the plasma over an oscillation period. ///

1.19 Consider a hypothetical situation in which all the electrons in a thin slab are displaced a small distance x_0 in the $\hat{\mathbf{e}}_{\rm x}$ direction. Show that the electric field induced by this displacement is given by $\mathbf{E} = (n_0 e/\epsilon_0)\, x\, \hat{\mathbf{e}}_x$ in the region where the electrons are displaced. Then, show from Newton's second law that this force causes the position of the slab of displaced electrons to oscillate at the electron plasma frequency. //

1.20 Taking account of plasma sheath effects, sketch the spatial variation of the potential $\Phi(x)$ between the plates of a capacitor filled with plasma assuming the capacitor has a potential $\Phi \sim 3\ T_e/e$ applied across it. Next, consider a case where an oscillating potential $\Phi = \Phi_0 \sin\omega t$ is applied across the plasma capacitor with $\Phi_0 = 10\ T_e/e$ and $\omega = \omega_{pe}/10$. If the capacitor plate separation is L ($>> \lambda_D$), how large is the electric field component oscillating at frequency ω in the body of the plasma? //

1.21 Show that for a one-dimensional wave perturbation in a plasma with $\tilde{\mathbf{E}}(x,t) = \hat{E}\,\hat{\mathbf{e}}_{\rm x}\sin(kx-\omega t)$ the nonlinear terms in (1.31) are negligible in (1.32) when the wave-induced velocity "jitter" in the particle motion, $\tilde{v}_{\rm jitter} \equiv q\hat{E}/m\omega$, is small compared to the wave phase speed ω/k, or alternatively when $k\tilde{x}_{\rm jitter} << 1$. //

1.22 Consider the propagation of ion acoustic waves in a typical hollow cathode arc discharge composed of electrons and doubly charged Argon ions with $n_e = 10^{19}$ m^{-3}, $T_e = 10$ eV, $T_i = 1$ eV. Discuss why the conditions for propagation of ion acoustic waves are satisfied in this plasma. What is the ion acoustic speed in this plasma? Compare it to the speed of sound in air at the earth's surface. With what wavelength and phase speed will externally imposed waves with a frequency of 100 MHz propagate in this plasma? /

1.23 Show that the transverse electric field $\tilde{\mathbf{E}}_t$ induced by a small free current $\mathbf{J}_{\rm free}$ in a plasma is governed by the equation

$$\left(\nabla^2 - \frac{\omega_{pe}^2}{c^2}\right)\tilde{\mathbf{E}}_t - \frac{1}{c^2}\frac{\partial^2\tilde{\mathbf{E}}_t}{\partial t^2} = \mu_0\frac{\partial\mathbf{J}_{\rm free}}{\partial t}.$$

Then, show that a Green's function solution of this equation in an infinite, homogeneous plasma which satisfies this equation is

$$\tilde{\mathbf{E}}_t = -\frac{\mu_0}{4\pi}\int d^3x' \,\left[\frac{\partial\mathbf{J}_{\rm free}(\mathbf{x}',t')/\partial t'}{|\mathbf{x}-\mathbf{x}'|}\exp\left(-\frac{|\mathbf{x}-\mathbf{x}'|}{c/\omega_{pe}}\right)\right]_{ret}$$

in which the square bracket $[\;]_{ret}$ means that the time t' is to be evaluated at the retarded time $t' = t - |\mathbf{x}-\mathbf{x}'|/c$. ///

1.24 Use the solution in the preceding problem to calculate the transverse electric field $\tilde{\mathbf{E}}_t$ caused by the current $q_t\mathbf{v}\delta[\mathbf{x}-\mathbf{x}(t)]$ produced by a nonrelativistic test particle moving along the trajectory $\mathbf{x} = \mathbf{x}(t)$ in a plasma. Show that this transverse electric field points in the direction of test particle motion. Also, show that for $|\mathbf{x}-\mathbf{x}(t)| < \lambda_D$ its magnitude is of order $v^2/c^2 << 1$ compared to the longitudinal electric field produced by the electrostatic potential ϕ_t in (1.10). ///

1.25 Plot the wave dispersion diagrams for electrostatic ion acoustic waves and electromagnetic plasma waves (i.e., Figs. 1.8 and 1.9) in the plasma described in Problem 1.22 on a single ω versus k diagram with approximately linear scales. Indicate in which regions of this diagram adiabatic and inertial responses for the electrons and ions are applicable. /

1.26 A 140 GHz microwave interferometer set up across a 30 cm thick column of plasma measures a phase shift of 240°. What is the "line-average" plasma density in the column? /

1.27 Amateur radio operators routinely communicate via shortwave radio over long distances around the earth. Since communication by direct line of sight is not possible because of the curvature of the earth's surface, the waves must be reflected from the ionosphere above the earth's surface. What frequency range corresponds to the 10 to 40 meter free space wavelength range used by amateur radio operators for these communications? What is the minimum electron density and height of the ionosphere above the earth's surface for single-bounce communications over the approximately 6000 km from the United States to Western Europe? /

1.28 During reentry of satellites into the earth's upper atmosphere, microwave communications in the 300 MHz frequency range are "blacked out" by the plasma formed in the heated air around the satellite. How high must the plasma density be around the satellite and how thick must the plasma be to cause the communications blackout? /

1.29 In one type of inertial fusion experiment, intense light from a laser is shined on a frozen hydrogen pellet. As the laser light is absorbed it heats up the pellet and produces a plasma on its surface. Light from a Neodynium glass laser ($\lambda = 1.06\ \mu$m) is ultimately observed to be reflected from the pellet. How high must the density of "free" electrons be in the plasma around the pellet? Compare this density to the original solid density of the pellet. How thick must the layer of free electrons be to reflect (or refract) the light waves? Compare this length to a typical pellet radius of 3 mm. /

1.30 In plasma processing of materials for the semiconductor industry an inert, low pressure gas is partially ionized by radiofrequency waves in a vacuum chamber. Consider a case where the initial gas is Argon at a pressure of 10^{-4} mm Hg (a 760 mm column of mercury corresponds to atmospheric pressure), the electron density is 10^7 cm^{-3}, the electron temperature is 3 eV and the temperature of the singly charged Argon ions is 0.1 eV. What is the degree of ionization in this gas? Estimate the electron-neutral collision frequency ν_{en} using an electron-neutral cross-section of $10^3 \pi a_0^2$ where a_0 is the Bohr radius. Does this medium satisfy all the criteria for being a plasma? How large must it be to satisfy the length criterion? /

1.31 An oscillating potential of 3 volts at a frequency of 1 MHz is applied to a probe inserted into the plasma described in Problem 1.9. Over what distance ranges from the probe can adiabatic or inertial responses be used for the electrons and for the ions in this plasma? /

1.32 What is the dielectric "constant" for externally imposed waves with a frequency of 1 MHz and a wavelength of 5 cm in the plasma described in problem 1.9? What would the dielectric "constant" be if the wavelength was increased to 500 cm? /